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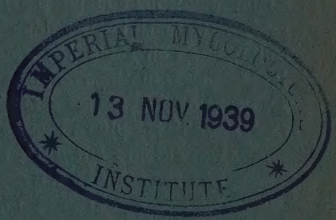

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MATHÉMATIQUES

SUR LES COUPLES DE CONGRUENCES STRATIFIABLES, LIÉS À LA CONGRUENCE DE BIANCHI

Par S. BACHVALOFF

(Présenté par N. N. Lusin, de l'Académie, le 10. VI. 1939)

Deux congruences L_1 et L_2 ayant pour rayons correspondants l_1 et l_2 sont appelées couple de congruences stratifiables dans deux sens, si l'on peut adjoindre une famille de surfaces $\sum_1(\infty^1)$ et une famille de surfaces $\sum_2(\infty^1)$ de manière que

1) les plans tangents aux surfaces $\sum_1$ le long du rayon l_1 passent par l_2 et

2) les plans tangents aux surfaces $\sum_2$ le long du rayon l_2 passent par l_1.

Soit L_3 la congruence des perpendiculaires l_3 communes aux rayons l_1, l_2 et N_1, N_2 les points d'intersection des rayons l_1, l_2 avec l_3.

Dans la présente Note nous considérons le cas où les surfaces N_1, N_2 appartiennent respectivement aux familles $\sum_1$ et $\sum_2$.

Le problème et résolu par la méthode du triangle mobile. Prenons pour sommet du trièdre T le milieu M du segment N_1N_2 et pour axes des droites parallèles aux rayons l_1, l_2, l_3. Soient $\bar{I}_1, \bar{I}_2, \bar{I}_3$ les vecteurs unitaires des axes du trièdre. Un déplacement du trièdre ainsi adjoint au couple des congruences stratifiables L_1, L_2 est défini par le système d'équations suivant [1]:

$$d\bar{M} = \omega_1\bar{I}_1 + \omega_2\bar{I}_2 + \omega_3\bar{I}_3 \tag{1}$$

$$d\bar{I}_j = \omega_{j1}\bar{I}_1 + \omega_{j2}\bar{I}_2 + \omega_{j3}\bar{I}_3 \tag{2}$$

$$\omega_{11} = -p\omega_{12}, \quad \omega_{22} = -p\omega_{21}, \quad \omega_{33} = 0 \tag{3}$$

$$dp = (1-p^2)(\omega_{12}+\omega_{21}), \quad \omega_{31}+p\omega_{32}+\omega_{13}=0, \quad p\omega_{31}+\omega_{32}+\omega_{23}=0 \tag{4}$$

$$[\omega_3 - da, \omega_{13}] + [2a\omega_{12}, \omega_{23}] = 0, \quad [\omega_3 + da, \omega_{23}] + [-2a\omega_{21}, \omega_{13}] = 0 \tag{5}$$

$$[\omega_1 - a\omega_{31}, \omega_3 + da] + [\omega_2 + a\omega_{32}, 2a\omega_{21}] = 0$$

$$[\omega_2 + a\omega_{32}, \omega_3 - da] + [\omega_1 - a\omega_{31}, -2a\omega_{12}] = 0 \tag{6}$$

$$[\omega_1, \omega_{13}] - [\omega_2, \omega_{23}] = 0 \tag{7}$$

$$[da, \omega_3] + 2a^2[\omega_{12}, \omega_{21}] = 0 \tag{8}$$

où

$$2a = N_1N_2 \text{ et } p = \bar{I}_1\bar{I}_2. \tag{9}$$

Il suit de (9) que

$$\bar{N}_1 = \bar{M} + a\bar{I}_3, \quad \bar{N}_2 = \bar{M} - a\bar{I}_3. \tag{10}$$

On en déduit que

$$\left.\begin{aligned} d\overline{N}_1 &= (\omega_1 + a\omega_{31})\,\overline{I}_1 + (\omega_2 + a\omega_{32})\,\overline{I}_2 + (\omega_3 + da)\,\overline{I}_3 \\ d\overline{N}_2 &= (\omega_1 - a\omega_{31})\,\overline{I}_1 + (\omega_2 - a\omega_{32})\,\overline{I}_2 + (\omega_3 - da)\,\overline{I}_3 \end{aligned}\right\} \qquad (11)$$

En vertu de l'hypothère faite les plans tangents aux surfaces N_1, N_2 contiennent respectivement les couples de vecteurs $(\overline{I}_2, \overline{I}_3)$ et $(\overline{I}_1, \overline{I}_3)$. Donc, dans les équations (11)

$$\omega_1 + a\omega_{31} \equiv 0, \quad \omega_2 + a\omega_{32} \equiv 0. \qquad (12)$$

En différentiant extérieurement ω_{31}, ω_{32} on obtient de (12)

$$[\omega_1, \omega_3 + da] + [\omega_2, 2a\omega_{21}] = 0, \quad [\omega_1, -2a\omega_{12}] + [\omega_2, \omega_3 - da] = 0. \qquad (13)$$

En appliquant aux équations (13) le lemme de Cartan [2] on obtient

$$\left.\begin{aligned} \omega_3 + da &= x\omega_1 + y\omega_2, \quad -2a\omega_{12} = s\omega_1 + t\omega_2 \\ 2a\omega_{21} &= y\omega_1 + z\omega_2, \quad \omega_3 - da = t\omega_1 + r\omega_2. \end{aligned}\right\} \qquad (14)$$

Il suit de (4, 12) que

$$\omega_{13} = \frac{\omega_1 - p\omega_2}{a}, \quad \omega_{23} = \frac{p\omega_1 - \omega_2}{a}. \qquad (15)$$

En substituant les valeurs $\omega_3 + da$, $\omega_3 - da$, ... dans les équations (5, 6, 7, 8) nous obtenons

$$r = s, \quad z = x.$$

Ainsi le système des équations (14) prend la forme

$$\left.\begin{aligned} \omega_3 + da &= x\omega_1 + y\omega_2, \quad -2a\omega_{12} = s\omega_1 + t\omega_2 \\ 2a\omega_{21} &= y\omega_1 - x\omega_2, \quad \omega_3 - da = t\omega_1 + s\omega_2 \end{aligned}\right\} \qquad (14')$$

En différentiant respectivement le système d'équations (3, 4, 12, 15, 14) nous obtenons

$$\left.\begin{aligned} [dx\omega_1] + [dy\omega_2] &= A\,[\omega_1, \omega_2] \\ [dy\omega_1] + [dx\omega_2] &= B\,[\omega_1, \omega_2] \\ [ds\omega_1] + [dt\omega_2] &= C\,[\omega_1, \omega_2] \\ [dt\omega_1] + [ds\omega_2] &= D\,[\omega_1, \omega_2] \end{aligned}\right\} \qquad (16)$$

où A, B, C, D sont des fonctions qui dépendent de x, y, s, t, p, a. Comme

$$\Delta = (\omega_1^2 - \omega_2^2)^2 \not\equiv 0$$

la solution du problème posé existe et dépend de 4 fonctions arbitraires d'un argument [2].

Démontrons que la congruence I_3 est une congruence de Bianchi. Une congruence est nommée congruence de Bianchi [3] si

a) L_3 est une congruence W,

b) les courbures de Gauss aux points correspondants N_1, N_2 des surfaces focales sont égales.

Comme $[\overline{I}_2\overline{I}_3]$, $[\overline{I}_1\overline{I}_3]$ sont les vecteurs unitaires des normales aux surfaces N_1, N_2, les secondes formes quadratiques φ_1, φ_2 de ces surfaces

ont la forme

$$\left.\begin{aligned}\varphi_1 &= d^2\bar{N}_1\,[\bar{I}_2\bar{I}_3] = (\omega_2^2-\omega_1^2)\,\frac{x}{a}\,\sqrt{1-p^2}\\ \varphi_2 &= d^2\bar{N}_2\,[\bar{I}_1\bar{I}_3] = (\omega_2^2-\omega_1^2)\,\frac{s}{a}\,\sqrt{1-p^2}.\end{aligned}\right\} \qquad (17)$$

La comparaison de ces formes montre que sur les surfaces N_1, N_2 les lignes asymptotiques sont correspondantes.

On trouve des équations (11, 12, 14) les premières formes quadratiques des surfaces

$$\left.\begin{aligned}ds_1^2 &= d\bar{N}_1^2 = 4\omega_2^2 + (x\omega_1 + y\omega_2)^2\\ ds_2^2 &= d\bar{N}_2^2 = 4\omega_1^2 + (t\omega_1 + s\omega_2)^2.\end{aligned}\right\} \qquad (18)$$

Il suit des équations (17, 18) que les courbures de Gauss K_1, K_2 aux points N_1, N_2 sont égales et

$$K_1 = K_2 = -\frac{1-p^2}{4a^2}.$$

Il est facile à voir la congruence L_3 est une congruence de Bianchi arbitraire dont les lignes asymptotiques des surfaces focales sont réelles.

Comme la congruence pseudo-sphérique présente un cas particulier d'une congruence de Bianchi, nos raisonnements s'appliquent à une congruence pseudo-sphérique.

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MATHÉMATIQUES

SUR UN PROBLÈME DE M. CARLEMAN

Par M. KELDYCH et M. LAVRENTIEFF

(Présenté par S. L. Soboleff, de l'Académie, le 10. IV. 1939)

1. Position du problème. Considérons dans le plan de la variable complexe z un continu E qui contient le point à l'infini $z=\infty$. Nous dirons que le continu E est un continu de Carleman si, quelle que soit la fonction $f(z)$ définie et continue sur $E\,(z<\infty)$ et quelle que soit la fonction continue $\varepsilon(r)$, $r\geqslant 0$, on peut trouver une fonction entière $F(z)$ telle que

$$|f(z)-F(z)|<\varepsilon(|z|). \tag{1}$$

Carleman [1] a démontré que l'axe réel est un continu de Carleman et c'est aussi lui qui a démontré qu'au lieu d'une droite on peut prendre un système fini quelconque de courbes rectifiables de Jordan sans points multiples contenant le point a l'infini ∞ et deux à deux sans point commun dans la partie finie du plan.

A. Roth [2] a démontré qu'un continu partout non-dense formé par les rayons $\arg z=\text{const}$, $|z|\geqslant k\geqslant 0$ est toujours un continu de Carleman.

Dans la présente Note nous allons indiquer les conditions nécessaires et suffisantes pour qu'un continu E soit un continu de Carleman.

2. Théorème. *Pour qu'un continu E soit un continu de Carleman, il est nécessaire et suffisant que*

1) *E soit partout non-dense;*

2) *il existe une fonction croissante $r(t)$, $\lim\limits_{t\to\infty} r(t)=\infty$, telle que pour chaque point z situé hors de E il existe une courbe de Jordan issue du point z, allant vers ∞ et située hors de E et hors du cercle $|\zeta|<r(|z|)$.*

3. Nécessité de la condition. La nécessité de la première condition est évidente. Supposons maintenant qu'un continu E_0 partout non-dense ne vérifie pas la condition 2. Dans ce cas le complémentaire de E_0 n'est par convexe et l'un des domaines D qui forment ce complémentaire est borné, ou bien il existe un nombre $r_0>0$ et une suite de points $z_1, z_2, \ldots, z_n, \ldots$ tels que chaque ligne joignant le point z_k au point $z=\infty$ et située hors de E_0 coupe la circonférence $|z|=r_0$. Dans le premier cas nous désignons par z_0 un point arbitraire du domaine D et nous posons $f(z)=\dfrac{1}{z-z_0}$; si $\varepsilon(r)$ est assez petit, il n'existe pas de fonction entière $F(z)$ vérifiant l'inégalité (1) sur la frontière du domain D. Donc, E_0 ne peut pas être un continu de Carleman. Dans le second cas designons par D_n le plus grand domaine contenant le point z_n et situé

hors de E_0 et hors du cercle $|z| \leqslant r_0$. En vertu de la condition imposée, tous les domaines D_n sont finis. En choisissant de la suite $z_1, z_2, \ldots, z_n, \ldots$ une certaine suite partielle, nous pouvons toujours arriver à se que les domaines D_n soient deux à deux sans point commun et que les points frontières ζ_n du domaine D_n dont la distance de l'origine $z=0$ soit maximale vérifient l'inégalité $|\zeta_{n+1}| > 2|\zeta_n| > 3r_0$.

Cela étant, fixons dans chaque domaine D_n un point a_n, $|a_n| = 2r_0$, et sur la frontière de D_n un ensemble e_n de points situés dans le cercle $|z-\zeta_n| \leqslant r_0$. Faisons la représentation conforme du domaine D_n sur le cercle $|w| < 1$ de telle manière qu'au point a_n corresponde le point $w=0$ et désignons par λ_n la mesure de l'ensemble des points de la circonférence $|w|=1$ correspondant à l'ensemble e_n. Considérons maintenant sur E_0 une fonction arbitraire réelle et continue $f(z)$ jouissant des propriétés suivantes: 1) $f(z) > 0$ et 2) $f(z) > \frac{2\pi n}{\lambda_n}$ aux points de l'ensemble e_n. Il est facile à voir que si la fonction $\varepsilon(r)$ est bornée, il n'existe pas de fonction entière $F(z)$ vérifiant (1). En effet, supposons qu'une telle fonction existe et désignons par $P(z)$ sa partie réelle. Nous avons

$$|f(z) - P(z)| < C = \text{const}$$

en tous les points de E_0. Donc, en désignant par m la valeur minimale de $P(z)$ pour $|z| = r_0$, on a $P(a_n) > n - C + m$, $n = 1, 2, 3, \ldots$ ce qui est impossible puisque pour $|z| = |a_n| = 2r_0$ la fonction $P(z)$ est bornée.

Suffisance de la condition. Supposons maintenant que la continu E vérifie les conditions du théorème. Fixons arbitrairement un nombre R et soit E_R l'ensemble des points de E appartenant au cercle $|z| \leqslant R^{(1)}$, où $R^{(1)}$ est la racine de l'équation $r(R^{(1)}) = 2R$. Désignons par D_R le plus grand domaine connexe contenant le point $z=\infty$ et ne contenant aucun point de E_R et aucun point du cercle $|z| < r = r\left(\frac{1}{2}R\right)$. Soit E_R^* l'ensemble complémentaire au domaine D_R. L'ensemble E_R^* contient l'ensemble E_R ainsi que le cercle $|z| \leqslant r$; d'ailleurs, en vertu de la condition 2) du théorème sur la circonférence $|z| = \rho$, $\frac{1}{2}R < \rho$, les ensembles E_R et E_R^* coïncident. Donc, comme E_R^* est partout non-dense, il existe toujours un nombre $\rho_0 < R$ aussi proche de R que l'on veut et tel que l'ensemble des points E_R^* situés sur $|z| = \rho_0$ est partout non-dense sur cette circonférence. En vertu d'un lemme de M. Lavrentieff [3] il en suit que pour chaque nombre ν il existe un polynôme $\pi(z, R, \nu)$ jouissant des propriétés suivantes:

1) $|\pi(z, R, \nu)| < 2$ en tous les points de E_R^*,

2) $|\pi(z, R, \nu)| < \nu$ en tous les points de E_R^* pour $|z| \leqslant \frac{R}{2}$,

3) $|\pi(z, R, \nu) - 1| < \nu$ en tous les points de E_R^* pour $|z| \geqslant R$.

Démontrons maintenant que le continu considéré est un continu de Carleman. A cet effet prenons une fonction arbitraire $f(z)$, définie et continue sur E, et une fonction arbitraire positive $\varepsilon(r)$ et démontrons l'existence d'une fonction entière $F(z)$ vérifiant (1). Sans restreindre la généralité, nous pouvons, évidemment, supposer que $\varepsilon(r)$ décroît et tend vers zéro avec $\frac{1}{r}$. Tout d'abord, en vertu d'un théorème de M. Lavrentieff [3] on peut construire un polynôme $P(z)$ tel que

$$|f(z) - P_0(z)| \leqslant \frac{1}{4}\varepsilon(R_0), \quad R_0 > 0$$

en tous les points de E appartenant au cercle $|z| \leqslant R_0$. En vertu du même théorème il existe un polynôme $P_1^{(1)}(z)$ tel que

$$|f(z) - P_0(z) - P_1^{(1)}(z)| < \frac{1}{4}\varepsilon(R_1),\ r(R_1) = 2R_0$$

en tous les points de l'ensemble E_{R_0}. Posons $P_1(z) = P_1^{(1)}(z)\,\pi(z, R_0, \nu_1)$. Alors le nombre ν_1 étant choisi assez petit pour que l'on ait pour $|z| \leqslant r_0 = r(R_0)$ l'inégalité $|P_1(z)| < \frac{1}{2}$, nous aurons en tous les points de E_R pour $|z| < R_0$ l'inégalité

$$|f(z) - P_0(z) - P_1(z)| < \frac{1}{2}\varepsilon(R_0)$$

et en tous les points de E_R pour $|z| \geqslant R_0$ l'inégalité

$$|f(z) - P_0(z) - P_1(z)| < \frac{1}{2}\varepsilon(R_1).$$

Supposons que nous avons déjà défini les nombres $R_0, R_1, \ldots R_n$ et les polynômes $P_0(z), P_1(z), \ldots P_n(z)$ de telle manière que

$$|P_n(z)| < \frac{1}{2^n} \text{ pour } |z| < r_{n-1} = r(R_{n-1}), \tag{2}$$

$$|f(z) - P_0(z) - P_1(z) - \ldots - P_n(z)| < \frac{1}{2}\varepsilon(R_k) \text{ pour } R_{k-1} < |z| < R_k, \tag{3}$$

$$k = 0, 1, \ldots n,\ R_{-1} = 0$$

déterminons alors le nombre R_{n+1} et le polynôme $P_{n+1}(z)$. Posons

$$r(R_{n+1}) = r(R_n).$$

Construisons un polynôme $P_{n+1}^{(1)}(z)$ de telle manière qu'en tous les points de l'ensemble E_{R_n} on ait

$$|f(z) - P_0(z) - P_1(z) - \ldots - P_n(z) - P_{n+1}^{(1)}(z)| < \frac{1}{2^{n+2}}\varepsilon(R_{n+1}).$$

On obtiendra le polynôme cherché $P_{n+1}(z)$ en multipliant le polynôme $P_{n+1}^{(1)}$ par le polynôme π:

$$P_{n+1}(z) = P_{n+1}^{(1)}(z)\,\pi(z, R_n, \nu_{n+1})$$

où le nombre ν_{n+1} est choisi assez petit pour que le polynôme construit $P_{n+1}(z)$ jouisse des propriétés (2) et (3) (où l'on remplace, évidemment, n par $n+1$).

Nous obtenons ainsi une suite de polynômes $P_0(z), P_1(z), \ldots, P_n(z), \ldots$

En vertu de (2) la série $\sum_{n=0}^{\infty} P_n(z)$ formée de ces polynômes converge uniformément en chaque partie finie du plan, donc sa somme est une fonction entière

$$F(z) = \sum_{n=0}^{\infty} P_n(z).$$

D'ailleurs, en vertu de la construction même, on a en chaque point de E l'inégalité

$$|f(z) - F(z)| < \varepsilon(|z|).$$

Donc, le théorème est complètement démontré.

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MATHÉMATIQUES

SUR LES OPÉRATIONS LINÉAIRES TRANSFORMANT UN CERTAIN ENSEMBLE CONIQUE EN LUI-MÊME

Par M. KREIN

(Présenté par A. N. Kolmogoroff, de l'Académie, le 16. IV. 1939)

Dans tout ce qui suit nous adoptons la terminologie du livre connu de M. Banach (1).

Soient E un espace linéaire normé et $K \subset E$ — un ensemble possédant les propriétés suivantes:

1. Si $x \in K$, on a $\lambda x \in K$ pour $\lambda > 0$.
2. Si $x \in K$ et $y \in K$, on a $x+y \in K$.
3. Si $x \in K$, on a $-x \in K$.
4. K est un ensemble ouvert.

En vertu de ces propriétés l'ensemble K présente l'intérieur d'un corps convexe «conique» dont le «sommet» est au point $x=0$. Par conséquent, d'après le théorème d'Ascoli-Mazur (2), on peut faire passer par le point $x=0$ un plan d'appui de K, c'est-à-dire il existe une fonctionnelle linéaire $f(x)$ telle que

$$f(x) > 0 \text{ pour } x \in K. \tag{1}$$

L'ensemble de toutes les fonctionnelles linéaires f satisfaisant à la condition (1) sera désigné par K^*. Il est aisé de voir pue l'ensemble $K^* \subset E^*$ (E^* désigne l'espace conjugué à E), possède les propriétés 1, 2, 3 et ne jouit pas, en général, de la propriété 4.

L'inégalité (1) peut être précisée, à savoir: si f appartient à K^* et le point u avec toute une sphère $S(u, \rho)$ de rayon ρ appartient á K, on a

$$f(u) > \rho |f|. \tag{2}$$

Nous allons établir le théorème suivant*.

Théorème 1. *Si une opération linéaire A ($AE \subset E$) transforme l'ensemble K en lui-même ($AK \subset K$), l'opération conjuguée A^* ($A^* E^* \subset E$)* a dans K^* un vecteur caractéristique* (*Eigenvektor*).

$$A^* f_0 = \lambda_0 f_0 \quad (\lambda_0 > 0).$$

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* Dans le premier énoncé du théorème 1 l'auteur imposait à l'ensemble K encore une condition outre les conditions 1—4. C'est à M. Rutmann que l'auteur est redevable de la remarque que tous ses raisonnements n'utilisent que les propriétés 1—4 de l'ensemble K.

Demonstration. Il est aisé de voir que si $AK \subset K$, alors $A^*K^* \subset K^*$.

Supposons d'abord que l'espace E est séparable. Soit $\{x_n\}$ une suite dense dans la sphère $\|x\| \leqslant 1$. Introduisons dans l'espace E^* une nouvelle norme

$$|f|_1 = \sup_n \frac{|f(x_n)|}{n}$$

et, par conséquent, une nouvelle distance $\rho(g, f) = |g - f|$ $(g, f \in E^*)$.

Il est facile à voir (1) qu'avec cette définition de la distance, tout ensemble $H \subset E^*$ borné et faiblement fermé devient un ensemble compact, la convergence d'après cette distance y coïncidant avec le convergence faible.

Soit à présent $u \in K$ un élément arbitraire appartenant à K avec la sphère $S(u, \rho)$. Désignons par H l'ensemble de toutes les $f \in E^*$, satisfaisant aux deux conditions suivantes:

$$f(u) = 1, \quad f \in K^*.$$

En vertu de l'inégalité (2) l'ensemble H est borné; il est aussi évident que H est faiblement fermé. Par conséquent, H est un ensemble compact (par rapport à la nouvelle distance) et évidemment convexe.

Envisageons d'autre part l'opération Bf definie par l'égalité:

$$Bf = \frac{A^*f}{f(Au)}.$$

Si $g = Bf$, on a

$$g(x) = \frac{f(Ax)}{f(Au)}. \tag{3}$$

En posant ici $x = u$, on trouve que $g(u) = 1$; d'où $BH \subset H$. D'autre part, si $f \subset K^*$, on a

$$f(Au) > \sigma |f|, \tag{4}$$

où $\sigma > 0$ est choisie sous la seule condition, que la sphère $S(v, \sigma)$ de centre $v = Au$ et de rayon σ appartienne à K. En outre si $f(u) = 1$, on a $|f| \cdot |u| \geqslant 1$. Ainsi

$$f(Au) > \frac{\sigma}{|u|} \text{ pour } f \in H.$$

De cette inégalité et de (3) on conclut facilement que l'opération Bf est faiblement continue sur H et par conséquent est continue sur H par rapport à la nouvelle distance.

Cela étant on peut appliquer à l'opération B considérée sur H le théorème connu de M. Schauder (3) sur l'existence d'un point invariant pour les transformations continues d'un ensemble convexe compact en lui-même. D'après ce théorème il existe un vecteur $f_0 \in H$ tel que $Bf_0 = f_0$, ou, ce qui est le même

$$A^*f_0 = \lambda_0 f_0, \tag{5}$$

où $\lambda_0 = f_0(Au) > 0$.

Ainsi le théorème est démontré pour le cas où l'espace E est séparable.

Passons maintenant au cas général. Observons d'abord qu'en vertu de la définition même de l'opération conjuguée A^* l'égalité (5) est équivalente à la suivante:

$$f_0(Ax - \lambda_0 x) = 0 \quad \text{pour tout } x \in E. \tag{6}$$

D'autre part, si $f_0 \in K^*$, la relation (6) en vertu de (1) implique

$$Ax - \lambda_0 x \overline{\in} K \text{ pour tout } x \in E. \qquad (7)$$

De plus, cette condition étant nécessaire est aussi suffisante pour qu'il existe un vecteur caractéristique $f_0 \in K^*$ correspondant au nombre λ_0. En effet, si le vecteur x parcourt E, le vecteur $y = Ax - \lambda_0 x$ parcourt un ensemble linéaire F. La condition (7) étant remplie, cet ensemble n'a pas de points communs avec K et par conséquent, d'après le théorème d'Ascoli-Mazur (2), il existe un plan d'appui à K passant par F, c'est-à-dire il existe une fonctionnelle $f_0 \in K^*$, telle que (5) a lieu.

Observons d'ailleurs que si à un nombre $\lambda_0 > 0$ correspond un vecteur caractéristique $f_0 \in K$ de l'opération A^*, on a alors

$$\sigma < \lambda_0 \leqslant \tau,$$

où le nombre σ est déterminé de la façon qu'on a indiquée pour (4) au moyen d'un vecteur $u \in K$ arbitrairement choisi, et $\tau = |A|$ est la norme de l'opération A.

Cela étant, faisons correspondre à chaque $x \in E$ l'ensemble $\mathfrak{M}_x$ formé de tous les nombres λ appartenant à l'intervalle ferme $< \sigma, \tau >$, pour lesquels

$$Ax - \lambda x \overline{\in} K.$$

Montrons que les éléments $x_i \in E$ $(i = 1, 2, \ldots, n)$ étant arbitrairement choisis, on a

$$\mathfrak{M}_{x_1} \mathfrak{M}_{x_2} \ldots \mathfrak{M}_{x_n} \neq 0. \qquad (8)$$

À cet effet considérons l'enveloppe linéaire E_1, de tous les éléments de la forme

$$A^k u \text{ ou } A^k x_i \; (i = 1, 2, \ldots, n;\; k = 0, 1, 2, \ldots,\; A^0 x = x)$$

et posons $K_1 = KE_1$. Il est aisé de voir que K_1, considéré dans E_1, possède toutes les propriétés $1-4°$. D'ailleurs, on a $AE_1 \subset E_1$ et $AK_1 \subset K_1$. L'espace E_1, étant séparable, le théorème à démontrer est applicable à l'opération A considérée dans E_1. En vertu de ce théorème il existe un nombre λ_1 $(\sigma < \lambda_1 \leqslant \tau)$, tel que

$$Ax - \lambda_1 x \overline{\in} K_1$$

pour tout $x \in E_1$ et, par suite, pour $x = x_i$ $(i = 1, 2, \ldots n)$, d'où résulte (8).

Comme chaque ensemble $\mathfrak{M}_x$ $(x \in E)$ est fermé, il suit de (8) que tous les ensembles $\mathfrak{M}_x$ ont un point commun λ_0 $(\sigma \leqslant \lambda_0 \leqslant \tau)$. Pour ce point λ_0 la condition (7) est remplie et, par conséquent, il lui correspond une fonctionnelle $f_0 \in K^*$ telle que $A^* f_0 = \lambda_0 f_0$.

Le théorème est dons démontré.

Observons que dans le cas particulier, où l'opération est complètement continue, le théorème 1 n'est pas nouveau et a été obtenu parmi d'autres théorèmes par M. A. Rutmann (4). Dans ce cas l'opération A elle-même possède un vecteur caractéristique (appartenant à K ou à sa borne, comme l'a montré M. A. Rutmann).

Pourtant dans le cas général le dernier fait peut ne pas avoir lieu. Prenons pour E, par exemple, l'espace $C(0,1)$ des fonctions continues $x(t)$ $(0 \leqslant t \leqslant 1)$ et pour K l'ensemble de toutes les fonctions positives $x(t) \in C(0,1)$. Posons $Ax(t) = p(t)\, x(t)$, où $p(t)$ $(0 \leqslant t \leqslant 1)$ est une fonction continue positive monotone quelconque. On a alors $AK \subset K$, et en

même temps l'opération A n'a evidemment aucun vecteur caractéristique dans $C(0,1)$.

A l'aide des raisonnements un peu plus compliqués on établit le

Théorème 2. *Soit G un ensémble d'opérations linéaires commutatives entre elles, et $G^* - l'ensemble$ de toutes les opérations conjuguées aux opérations de G. Si chaque $A \in G$ transforme K en lui-même ($AK \subset K$), les opérations $A^* \in G^*$ ont dans K^* un vecteur caractéristique commun f_0:*

$$A^* f_0 = \lambda_A f_0 \quad (\lambda_A > 0, \quad A \in G). \tag{9}$$

Notons le corollaire suivant du théorème 2.

Corollaire. *Les conditions du théorème 2 étant remplies, si toutes les operations $A \in G^*$ ont dans K un vecteur invariant x_0 commun*

$$Ax_0 = x_0 \quad (A \in G) \tag{10}$$

on a le même pour les opérations $A^* \in G$, c'est-à-dire il existe un vecteur $f_0 \in K^*$ tel que

$$A^* f_0 = f_0 \quad (A^* \in G^*). \tag{11}$$

En effet, en appliquant aux deux parties de l'égalité (10) la fonctionnelle f_0, dont on a énoncé l'existence dans le théorème 2, on trouve en vertu de (9) que $f_0(x_0) = f_0(Ax_0) = \lambda_A f_0(x_0)$. Comme $f_0(x_0) > 0$ (car $x_0 \in K$, $f_0 \in K^*$), on en conclut que $\lambda_A = 1$ et on arrive ainsi à (11).

Nous allons montrer que le corollaire du théorème 2 contient comme cas particulier un théorème de A. Markov [5].

Prenons dans ce but pour E l'espace de toutes les fonctions réelles bornées $x(q)$ définies sur un ensemble Q avec la définition habituelle de la norme:

$$\| x \| = \sup_{q \in Q} | x(q) |.$$

Prenons pour K l'ensemble de toutes les fonctions $x(q) \in E$, telles que

$$\inf_{q \in Q} x(q) > 0.$$

Soit Γ un ensemble des transformations univoques φ de l'ensemble Q en lui-même qui sont commutatives entre elles, c'est-à-dire

$$\varphi(\psi(q)) = \psi(\varphi(q)) \quad (\varphi, \psi \in \Gamma).$$

Chaque transformation $\varphi \in \Gamma$ génère une opération linéaire A_φ, où

$$A_\varphi x(q) = x(\varphi(q)) \; (= x_\varphi).$$

On a évidemment $A_\varphi K \subset K$, $A_\varphi A_\psi = A_\psi A_\varphi$ $(\varphi, \psi \in \Gamma)$ et en outre

$$A_\varphi 1 = 1 \quad (\varphi \in \Gamma).$$

Ainsi on peut appliquer à l'ensemble des opérations A_φ $(\varphi \in \Gamma)$ le corollaire du théorème 2 et on arrive alors à l'existence d'une fonctionnelle $f_0 \in K^*$ telle que $A_\varphi^* f_0 = f_0$ $(\varphi \in \Gamma)$ ou, ce qui est le même,

$$f_0(x_\varphi) = f_0(x) \qquad (x \in K, \quad \varphi \in \Gamma).$$

L'existence d'une felle fonctionnelle $f_0 \in K^*$ fait justement le contenu d'un théorème de A. Markov [5].

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MATHÉMATIQUES

SUR L'ÉQUIVALENCE DES FAMILLES D'ENSEMBLES MESURABLES *B*

Par G. OTCHAN

(Présenté par A. N. Kolmogoroff, de l'Académie, le 16. IV. 1939)

Dans tout ce qui suit nous désignons par R un espace métrique complet ayant une base dénombrable.

Deux familles d'ensembles de l'espace R seront nommées équivalentes si l'espace admet une transformation biunivoque en lui-même telle que chaque ensemble de la première famille se transforme en un ensemble de la seconde famille et réciproquement *.

E. Szpilrajn (1) a établi plusieurs théorèmes concernant l'équivalence et la non-équivalence pour deux familles dénombrables d'ensembles. Ainsi, par exemple, il a démontre que pour chaque n il existe une famille dénombrable d'ensembles projectifs de classe n qui n'est équivalente à aucune famille dénombrable d'ensembles projectifs de classes inférieures.

Il suit de là, en particulier, que la famille de tous les ensembles projectifs de classe n n'est équivalente à aucune famille d'ensembles projectifs de classes inferieures.

Dans cette Note, parmi d'autres résultats, on démontre une proposition analogue pour les classes d'ensembles mesurables B. La famille de tous les ensembles mesurables B de classe $F^\xi(G^\xi)$ n'est équivalente à aucune famille formée seulement d'ensembles de classes inférieures [voir pour les notations Hausdorff (2)]. Contrairement à ce qui a lieu pour les ensembles projectifs, ce résultat ne peut pas être obtenu par la considération des familles dénombrables d'ensembles, car il a été démontré par Szpilrajn qu'une famille dénombrable quelconque d'ensembles mesurables B est équivalente à une certaine famille d'ensembles, formée seulement d'ensembles qui sont F_σ et G_δ simultanément.

Lemme. *Soit $\mathfrak{M}$ une famille d'ensembles appartenant à R et telle que chaque ensemble ouvert de l'espace R appartient à la fermeture borélienne de la famille $\mathfrak{M}$ **. Si $\mathfrak{M}$ est équivalente à une famille $\mathfrak{N}$, formée seulement d'ensembles mesurables B, cette équivalence ne peut être effectuée qu'au moyen d'une transformation de R en lui-même parune fonction mesurable B.*

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* La transformation biunivoque dont il s'agit ici n'est assujettie à aucune restriction concernant la continuité, la mesurabilité, etc. Ainsi la notion elle-même d'equivalence de deux familles d'ensembles appartenant à un ensemble R appartient à la théorie des ensembles pure. Mais les résultats ulterieurs de cette Note sont essentiellement lies à la nature topologique précedemment indiquée de l'espace R.

** On appelle fermeture borelienne d'une famille d'ensembles la plus petite famille d'ensembles contenant la famille donnée et invariante par rapport aux opérations: addition d'ensembles en nombre fini ou en infinité dénombrable et prise du complémentaire (c'est-à-dire passage de E à $R-E$).

Démonstration. Soit $y=f(x)$ une transformation biunivoque de R en lui-même qui transforme $\mathfrak{M}$ en $\mathfrak{N}$ et $x=f^{-1}(y)$ — la transformation inverse. Soit ensuite $G_1, G_2, \ldots G_n \ldots$ un système complet de voisinages de l'espace et $T_n=f(G_n)$. Comme G_n appartient à la fermeture borélienne de $\mathfrak{M}$, les ensembles T_n appartiennent à la fermeture borélienne de $\mathfrak{N}$, donc ce sont des ensembles mesurables B. Désignons par α la borne supérieure des classes des ensembles T_n. Quel que soit l'ensemble ouvert G de l'espace R, il peut être présenté dans la forme $G=\sum_k G_{n_k}$. Donc, l'ensemble

$$T=f(G)=f\left(\sum_k G_{n_k}\right)=\sum_k T_{n_k}$$

est un ensemble mesurable B et de classe inférieure ou égale à $\alpha+1$. Autrement dit, dans la transformation $x=f^{-1}(y)$ les ensembles ouverts seront des images d'ensembles mesurables B des classes inférieures ou égales à $\alpha+1$. Mais cela signifie que la fonction f^{-1} est une fonction mesurable B. En vertu de ce que la transformation $y=f(x)$ est biunivoque, il en résulte que la fonction $f(x)$ est aussi mesurable B.

Définition. Une famille $\mathfrak{N}$ est appelée N-famille si elle jouit des propriétés suivantes: 1. Chaque ensemble ouvert de l'espace B appartient à la fermeture borélienne de la famille $\mathfrak{N}$. 2. Chaque ensemble parfait compact de l'espace R contient au moins un ensemble de la famille $\mathfrak{N}$.

Théorème 1. *Une N-famille $\mathfrak{N}$ dont tous les ensembles possèdent une certaine propriété K topologiquement invariante ne peut être équivalente à aucune famille $\mathfrak{M}$, formée seulement d'ensembles mesurables B et dont aucun ne jouit de la propriété K.*

Démonstration. Supposons, par impossible, qu'il existe une transformation biunivoque $y=f(x)$ de l'espace R en lui-même tel que $\mathfrak{N}$ soit transformé en $\mathfrak{M}$; alors, en vertu du lemme, la fonction f est mesurable B. Or, en vertu du théorème de Baire, il existe un ensemble G_δ dense dans R, où cette fonction mesurable B est continue. Mais alors cette fonction est encore continue sur un noyau parfait compact P de ce G_δ. Si une transformation biunivoque est effectuée par une fonction continue sur un ensemble compact, la fonction inverse est encore continue, donc la transformation $f(P)=Q$ est une homéomorphie. D'après l'hypothèse, il existe sur P un ensemble R de la famille $\mathfrak{N}$ jouissant de la propriété K. L'ensemble $S=f(R)$ jouit aussi de la propriété K et appartient à $\mathfrak{M}$, ce qui contredit à l'hypothèse faite. Cette contradiction démontre le théorème.

Nous dirons qu'un ensemble mesurable B appartient à $F^\xi(G^\xi)$ au sens strict, s'il est $F^\xi(G^\xi)$ sant être $G^\xi(F^\xi)$.

Nous dirons qu'un ensemble appartient à la classe $F^\xi(G^\xi)$ au sens restreint, si chacune de ses portions* non-vide est un ensemble $F^\xi(G^\xi)$ au sens strict. Pour $\xi\geqslant 2$ la propriété d'être $F^\xi(G^\xi)$ au sens strict ainsi qu'au sens restreint est topologiquement invariante. Ceci permet d'établir le

Corollaire I. *Si $\xi\geqslant 2$, la famille de tous les ensembles de classe $F^\xi(G^\xi)$ au sens restreint n'est équivalente à aucune famille d'ensembles mesurables B n'ayant avec la famille donnée aucun ensemble commun.*

Démonstration. Pour démontrer ce corollaire, il suffit d'établir que l'ensemble de tous les ensembles de classe F^ξ (resp. G^ξ) au sens

* On appelle portion d'un ensemble la partie commune de cet ensemble et d'un ensemble ouvert.

restreint forme une N-famille. Il est évident que la seconde condition imposée aux N-familles est remplie. En ce qui concerne la première condition, il est facile à voir que pour ξ pair (ξ impair) un ensemble ouvert quelconque peut être présenté comme une somme dénombrable d'ensembles F^ξ (G^ξ) au sens restreint et pour ξ impair (resp. ξ pair) même comme une somme de deux ensembles de classe F^ξ (resp. G^ξ) au sens restreint.

Corollaire II. *La famille de tous les ensembles fermés compacts n'est équivalente à aucune famille d'ensembles mesurables B dont aucun n'est formé et compact.*

Corollaire III. *La famille de tous les ensembles mesurables B et* 1) *de classe* I *au plus*, 2) *de classe* F^ξ ($\xi \geqslant 2$) *au sens strict*, 3) *de classe* G^ξ ($\xi \geqslant 2$) *au sens strict—n'est équivalente à aucune famille d'ensembles mesurables B dont aucun n'est respectivement un ensemble:* 1) *fermé compact*, 2) *de classe* F^ξ ($\xi \geqslant 2$) *au sens restreint*, 3) *de classe* G^ξ ($\xi \geqslant 2$) *au sens restreint.*

On en déduit en particulier le

Corollaire IV. *Pour* $\xi \geqslant 2$ *la famille de tous les ensembles* F^ξ (G^ξ) *n'est équivalente à aucune famille formée seulement d'ensembles* G^ξ (F^ξ).

Le corollaire prouve l'affirmation faite au commencement de cete Note.

Introduisons maintenant la notion de «condensation d'une famille».

On dit qu'une famille $\mathfrak{N}$ peut être condensée en sa partie $\mathfrak{M}$, s'il existe une transformation biunivoque de l'espace en lui-même telle que $\mathfrak{N}$ est transformé en sa partie $\mathfrak{M}$.

Appelons ensuite N-famille maximale ayant la propriété K le système—somme de toutes les N-familles jouissant de la propriété K. Nous aurons alors le théorème suivant:

Théorème II. *Si* $\mathfrak{N}$ *est une N-famille maximale ayant la propriété K et formée seulement d'ensembles mesurables B, elle ne peut pas être condensée en une famille* $\mathfrak{M}$ *faisant partie de* $\mathfrak{N}$ *et telle que* $\mathfrak{N}-\mathfrak{M}$ *est une N-famille.*

Démonstration. Supposons que $\mathfrak{N}$ est transformée en $\mathfrak{M}$ au moyen d'une fonction $y=f(x)$. En vertu du lemme cette fonction doit être mesurable B. Mais alors la fonction inverse $x=f^{-1}(y)$ (et encore mesurable B) transforme $\mathfrak{M}$ en $\mathfrak{N}$ et $\mathfrak{N}-\mathfrak{M}$ en une certaine famille d'ensembles mesurables B dont aucun ne jouit de la propriété K; ceci contredit au théorème I, puisque $\mathfrak{N}-\mathfrak{M}$ est une N-famille jouissant de la propriété K.

Les corollaires de ce théorème donnant des applications à des familles maximales particulières d'ensembles mesurables B sont évidents.

Remarquons encore qu'on déduit immédiatement du lemme le corollaire suivant:

Corollaire V. *La famille de tous les ensembles mesurables B n'est équivalente à aucune de ses parties aliquotes.*

On déduit du lemme et du corollaire V

Corollaire VI. *La famille de tous les ensembles A (ensembles de Souslin) n'est équivalente à aucune de ses parties aliquotes.*

En passant aux complémentaires on trouve

Corollaire VII. *La famille de tous les ensembles CA n'est équivalente à aucune de ses parties aliquotes.*

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MATHEMATICS

ON THE TAYLOR SERIES ASSOCIATED WITH AN INTEGRAL FUNCTION

By A. GELFOND, Corresponding Member of the Academy

In a number of questions involved in the function theory, e. g. in the problem of the magnitude of an integral function, in the question as to the distribution of singularities of a Taylor series on the circle of convergence (1), and in some problems of interpolation (2), an essential rôle is played by the Taylor series, associated, after Borel, with the given integral function of the first order and finite type. We call, after Borel, the function $f(z)$ associated with $F(z)$, if

$$F(z)=\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n \quad \text{and} \quad f(z)=\sum_{n=0}^{\infty}\frac{a_n}{z^{n+1}}. \tag{1}$$

It is known that if the type of the integral function of the first order $F(z)$ is σ, then the function $f(z)$ is regular outside the circle $|z|\leqslant\sigma$. In the present note I give one of the natural generalizations of the notion of the Taylor series associated with a given integral function. Suppose, in fact, that α is a real number. Then I call the function $f_\alpha(z)$ α-associated with the integral function $F(z)$ of the first order and the finite type σ, if

$$F(z)=\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n \quad \text{and} \quad f_\alpha(z)=\sum_{n=0}^{\infty}\frac{e^{+\frac{n(n-1)}{2}\alpha i}}{z^{n+1}}a_n. \tag{2}$$

Obviously, the function $f_\alpha(z)$ will be also regular everywhere outside the circle $|z|\leqslant\sigma$, since, according to the well known properties of integral functions of the fist order and finite type, $\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}=\sigma$. Between the functions $F(z)$ and $f_\alpha(z)$ there exists a connexion, similar to that existing between an integral function and the Taylor series associated with it after Borel. This connexion is given by the formulae:

$$\left.\begin{aligned} F(z)&=\frac{1}{2\pi i}\int_{\mathfrak{G}} u(z\xi)\,f_\alpha(\xi)\,d\xi,\\ F^{(n)}(z\xi^{i\alpha n})&=\frac{e^{\frac{n(n-1)}{2}\alpha i}}{2\pi i}\int_{\mathfrak{G}}\xi^n u(z\xi)\,f_\alpha(\xi)\,d\xi,\\ n&=1,2,\ldots, \end{aligned}\right\} \tag{3}$$

where c is the circumference $|\xi|=r$, $r>\sigma$, σ is the type of the integral function $F(z)$ and

$$u(z)=\sum_{n=0}^{\infty}\frac{e^{-\frac{n(n-1)}{2}\alpha i}}{n!}z^n. \tag{4}$$

The function $u(z)$ satisfies the functional equation $u'(z)=u(ze^{-i\alpha})$ and plays in this transformation the same rôle as the function e^z in Borel's transformation.

The formulae (3) may be easily proved by developing the function $f_\alpha(z)$ into a series and term-by-term integration of this series. With the help of the so defined function $f_\alpha(z)$ we may easily prove two theorems, one of which concerns the question on the structure of an integral function possessing given derivatives in a given sequence of points, and the other — the problem of determination, in the class of integral functions, of solutions of a type of functional equations.

Theorem 1. *If the integral function $F(z)$ of the order 1 and the type σ satisfies the conditions $F^{(n)}(e^{in\alpha})=0$, $n=0,1,2,\ldots$, then it must have the form*

$$\left.\begin{aligned} F(z)&=\sum_{k=0}^{m}\sum_{s=0}^{p_k-1}C_{k,s}z^s u(\lambda_k z),\\ u(z)&=\sum_{n=0}^{\infty}\frac{e^{-\frac{n(n-1)}{2}\alpha i}}{n!}z^n, \end{aligned}\right\} \tag{5}$$

where $C_{k,s}$ are constants, p_k is the multiplicity of the root λ_k of the function $u(z)$ and the summation with respect to k is extended over all roots of the function $u(z)$ not exceeding by modulus σ: $|\lambda_k|\leqslant\sigma$, $k=0,1,2,\ldots,m$, $|\lambda_{m+1}|>\sigma$.

From the conditions of our theorem and the formulae (3) it follows that for $F(z)$, $\sigma_1>\sigma$,

$$F^{(n)}(e^{\alpha n i})=\frac{e^{\frac{n(n-1)}{2}\alpha i}}{2\pi i}\int_{|\xi|=\sigma_1}\xi^n u(\xi)f_\alpha(\xi)\,d\xi=0,\qquad n=0,1,2,\ldots, \tag{6}$$

where $f_\alpha(\xi)$ is the function α-associated with $F(\xi)$. From the relations (6) it directly follows that $u(\xi)f_\alpha(\xi)=\varphi(\xi)$ is an integral function of ξ, since $u(\xi)$ is an integral function of ξ and $f_\alpha(\xi)$ is regular on the infinity. Consequently $f_\alpha(\xi)$, all singularities of which must lie within the circle $|\xi|\leqslant\sigma$, has the form:

$$f_\alpha(\xi)=\sum_{k=0}^{m}\frac{P_k(\xi)}{(\xi-\lambda_k)^{p_k}},\qquad P_k(\xi)=\sum_{s=0}^{p_k-1}A_{s,k}\xi^s, \tag{7}$$

where λ_k, $k=0,1,\ldots,m$, are all the roots of $u(z)=0$ lying in the circle $|z|\leqslant\sigma$ and $A_{s,k}$ are constants. Our theorem follows from (7), if we take into consideration the relations (3) between $F(z)$ and $f_\alpha(z)$.

Theorem 2. *If the integral function $F(z)$ of the first order and the type σ satisfies the functional equation with constant coefficients*

$$\sum_{k=0}^{\infty}b_k F^{(k)}(e^{i\alpha_n}z)=\Phi(z),\quad \overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}=\frac{1}{\tau}, \tag{8}$$

where Φ is an integral function of the order 1 and the type σ_1, $\sigma_1 \leqq \sigma < \tau$, then it may be represented in the form:

$$F(z) = \frac{1}{2\pi i} \int_{|\xi|=r} u(z\xi) \frac{\varphi_\alpha(\xi)}{v_\alpha(\xi)} d\xi + \sum_{k=0}^{m} \sum_{s=0}^{P_k - 1} C_{k,s}\, u(\lambda_k z) z^s, \tag{9}$$

where $\varphi_\alpha(z)$ is α-associated with $\Phi(z)$,

$$v_\alpha(\xi) = \sum_{k=0}^{\infty} e^{\frac{k(k-1)}{2}\alpha i} b_k \xi^k,$$

$u(z)$ was defined above [formula (5)], the outer sum on the right hand side of (9) is extended over all zeroes λ_k, of the multiplicity P_k of the function $v_\alpha(\xi)$ which lie in the circle $|\xi| \leqq \sigma$, $|\lambda_k| \leqq \sigma$, $k = 0, 1, \ldots, m$, $|\lambda_{m+1}| > \sigma$ and r is subjected to the inequalities $\sigma < r < |\lambda_{m+1}|$; $C_{k,s}$, $0 \leqq s \leqq P_k - 1$, $0 \leqq k \leqq m$, are constant numbers.

The formula (9) gives the general solution of the equation (8) in the class of integral functions of the first order and of the type not exceeding σ. The number of arbitrary constants contained in this general solution is equal to the number of zeroes of the function $v_\alpha(\xi)$ in the circle $|\xi| \leqq \sigma$, each zero being counted as many times as its multiplicity indicates.

On ground of formulae (3) the equation (8) may be written in the form

$$\frac{1}{2\pi i} \int_{|\xi|=R} u(z\xi) [v_\alpha(\xi) f_\alpha(\xi) - \varphi_\alpha(\xi)]\, d\xi = 0, \tag{10}$$

where $f_\alpha(\xi)$ is α-associated with $F(\xi)$, $v_\alpha(\xi)$ and $\varphi_\alpha(\xi)$ are defined in the conditions of the theorem and R is an arbitrary positive number subjected to the inequalities $\sigma < R < \tau$.

Since the equation (10) must hold for all values of z and all singularities of the functions $\varphi_\alpha(z)$ and $f_\alpha(z)$ must lie within the circle $|z| \leqq \sigma$, the function $v_\alpha(\xi) f_\alpha(\xi) - \varphi_\alpha(\xi)$ must be regular in the circle $|\xi| \leqq \tau$ and, consequently, $f_\alpha(\xi)$ has the form

$$f_\alpha(\xi) = g(\xi) + \sum_{k=0}^{m} \frac{P_k(\xi)}{(\xi - \lambda_k)^{p_k}}, \quad P_k(\xi) = \sum_{s=0}^{p_k - 1} A_{k,s} \xi^s, \tag{11}$$

where the sum is extended over all zeroes λ_k of multiplicity P_k of the function $v_\alpha(\xi)$ in the circle $|\xi| \leqq \sigma$, $|\lambda_k| < \sigma$, $k = 0, 1, \ldots, m$, $|\lambda_{m+1}| > \sigma$, $C_{k,s}$ are constant numbers and $g(\xi)$ is the principal part of the Laurent series of the function $\frac{\varphi_\alpha(\xi)}{v_\alpha(\xi)}$ in the ring $\sigma < |\xi| < |\lambda_{m+1}|$.

From the representation (11) of the function $f_\alpha(\xi)$ and the formulae (3), giving the connexion between $F(z)$ and $f_\alpha(z)$, immediately follows our theorem.

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THÉORIE DE L'ÉLASTICITÉ

ÉTUDE D'UN CAS D'ACTION NON-SYMÉTRIQUE D'UN POINÇON CIRCULAIRE APPLIQUÉ À UN CORPS ÉLASTIQUE D'ÉPAISSEUR INFINIE ET LIMITÉ PAR UN PLAN

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La solution du problème de l'action symétrique du poinçon a été donnée depuis longtemps par Boussinesq [1] [voir aussi [2]].

Si l'on choisit pour plan des x, y d'un système d'axes rectangulaires le plan qui limite le demi-espace élastique on peut, dans le cas d'absence des efforts tangentiels le long de ce plan, exprimer les déplacements u, v, w au moyen de quatre fonctions harmoniques $\varphi_1, \varphi_2, \varphi_3$ et ψ (désignant par G le module de rigidité et par m l'inverse du coefficient de Poisson) par les expressions suivantes:

$$u = \varphi_1 + x\frac{\partial \psi}{\partial x}, \quad v = \varphi_2 + y\frac{\partial \psi}{\partial y}, \quad w = \varphi_3 + z\frac{\partial \psi}{\partial z} \tag{1}$$

$$\psi = -\frac{m\varphi_3}{2(m-1)}, \quad \frac{\partial \varphi_1}{\partial z} = -\frac{m-2}{2(m-1)}\frac{\partial \varphi_3}{\partial x}, \quad \frac{\partial \varphi_2}{\partial z} = -\frac{m-2}{2(m-1)}\frac{\partial \varphi_3}{\partial y}. \tag{2}$$

On obtient moyennant les valeurs (1), (2) les expressions

$$Z_z = G\left\{\frac{m}{m-1}\frac{\partial \varphi_3}{\partial z} + 2z\frac{\partial^2 \psi}{\partial z^2}\right\} = G\left\{\frac{m}{m-1}\left(\frac{\partial \varphi_3}{\partial z} - z\frac{\partial^2 \varphi_3}{\partial z^2}\right)\right\} \tag{3}$$

$$w = \varphi_3 + z\frac{\partial \psi}{\partial z} = \varphi_3 - z\frac{m}{2(m-1)}\frac{\partial \varphi_3}{\partial z}. \tag{4}$$

Tout revient donc en général pour une action symétrique ou non du poinçon à la recherche d'une fonction harmonique φ_3.

1. Considérons le cas où le poinçon est soutenu sous l'action d'un seul moment M de forces dirigé suivant l'axe des y. Si R est le rayon du poinçon circulaire, les conditions aux limites sur le plan $z = 0$, qui sont nécessaires pour pouvoir déterminer φ_3 de notre problème, peuvent s'écrire sous la forme:

$$\text{pour } z = 0 \begin{cases} w = \gamma x = \gamma r\cos\vartheta & \text{pour } r = \sqrt{x^2+y^2} \leqslant R \\ Z_z = 0 & \text{pour } r = \sqrt{x^2+y^2} > R. \end{cases} \tag{5}$$

Pour φ_3 nous aurons donc les conditions

$$\text{pour } z = 0 \begin{cases} \varphi_3 = \gamma r\cos\vartheta & \text{pour } r \leqslant R \\ \dfrac{\partial \varphi_3}{\partial z} = 0 & \text{pour } r > R \end{cases} \tag{6}$$

avec un certain coefficient γ angulaire que nous aurons à déterminer dans la suite. Construisons d'abord la fonction harmonique particulière nulle à l'infini avec un paramètre p arbitraire $e^{-pz}J_1(pr)\cos\vartheta$. Avec cette fonction et une fonction $A(p)$ indéterminée nous pouvons construire la solution qui satisferait les conditions aux limites. Il suffira de déterminer convenablement la fonction $A(p)$ dans l'expression de φ_3 sous forme d'intégrale définie:

$$\varphi_3 = \int_0^\infty A(p)\, e^{-pz} J_1(pr)\, dp \cos\vartheta. \tag{7}$$

Les conditions (6) prendront alors la forme

$$\left.\begin{aligned} &\int_0^\infty A(p)\, J_1(pr)\, dp \cos\vartheta = \gamma r \cos\vartheta \quad r \leqslant R \\ &\int_0^\infty -pA(p)\, J_1(pr)\, dp \cos\vartheta = 0 \quad r > R. \end{aligned}\right\} \tag{8}$$

La condition nécessaire à l'infini se trouvera également satisfaite. Posant:

$$r = R\rho, \quad p = \frac{t}{R}, \quad pA(p) = \gamma R f(t) \tag{9}$$

nous aurons les deux équations intégrales suivantes pour déterminer $f(t)$:

$$\left.\begin{aligned} &\int_0^\infty t^{-1} f(t)\, J_1(t\rho)\, dt = \rho \qquad (\rho \leqslant 1) \\ &\int_0^\infty f(t)\, J_1(t\rho)\, dt = 0 \qquad (\rho > 1). \end{aligned}\right\} \tag{10}$$

Mr Titchmarsh [3] et M-lle Busbridge [4] ont donné la solution d'un système un peu plus général d'équations:

$$\left.\begin{aligned} &\int_0^\infty t^\alpha J_\nu(t\rho)\, f(t)\, dt = \rho^\nu \qquad (\rho < 1) \\ &\int_0^\infty J_\nu(t\rho)\, f(t)\, dt = 0 \qquad (\rho > 1) \end{aligned}\right\} \tag{11}$$

sous la forme de l'expression très simple pour $f(t)$ suivante:

$$f(t) = \frac{\Gamma(\nu+1)}{\Gamma\left(\nu+1+\frac{1}{2}\alpha\right)} (2t)^{-\frac{1}{2}\alpha} J_{\nu+1+\frac{1}{2}\alpha}(t) \tag{12}$$

$$\left(\text{pour } \nu+2+\frac{1}{2}\alpha > 0\right).$$

Ainsi pour le cas considéré des équations (10) avec $\nu = 1$ et $\alpha = -1$ on aura

$$f(t) = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)} (2t)^{\frac{1}{2}} J_{\frac{3}{2}}(t) = \frac{4}{\pi}\left(\frac{\sin t}{t} - \cos t\right) \tag{13}$$

et par suite

$$A(p) = \frac{4\gamma}{\pi}\left(\frac{\sin pR}{p^2} - R\frac{\cos pR}{p}\right). \tag{14}$$

Calculons maintenant la dérivée

$$\frac{\partial \varphi_3}{\partial z} = \frac{4\gamma}{\pi}\int_0^\infty \left(-\frac{\sin pR}{p} + R\cos pR\right) e^{-pz} J_1(pr)\, dp \cos\vartheta =$$

$$= -\frac{4\gamma}{\pi}\left\{\int_0^\infty \frac{\sin pR}{p} e^{-pz} J_1(pr)\, dp - R\int_0^\infty \cos pR e^{-pz} J_1(pr)\, dp\right\}\cos\vartheta. \tag{15}$$

Le second terme du deuxième membre, celui qui contient cos pR, peut être calculé par l'intégrale connue de Beltrami [5]:

$$\int_0^\infty e^{-pz+ipR} J_1(pr)\, dp =$$

$$= \int_0^\infty e^{-pz}\cos pR J_1(pr)\, dp + i\int_0^\infty e^{-pz}\sin pR J_1(pr)\, dp = \frac{1}{r}\left(1 - \frac{z-iR}{Z-iB}\right) \tag{16}$$

dans laquelle les coordonnées elliptiques Z et B sont liées aux coordonnées cylindriques z et r par les relations:

$$r^2 + z^2 - R^2 = Z^2 - B^2,\ zR = ZB, \tag{17}$$

ou bien

$$\left.\begin{aligned} B &= \sqrt{\frac{1}{2}\left(\sqrt{(r^2+z^2-R^2)^2 + 4z^2R^2} - (r^2+z^2-R^2)\right)} \\ Z &= \sqrt{\frac{1}{2}\left(\sqrt{(r^2+z^2-R^2)^2 + 4z^2R^2} + (r^2+z^2-R^2)\right)}. \end{aligned}\right\} \tag{18}$$

Posant

$$Z = R\sigma, \quad B = R\mu \tag{19}$$

on obtiendra pour z et r les expressions

$$z = R\sigma\mu, \quad r = R\sqrt{(1-\mu^2)(1+\sigma^2)} \tag{20}$$
$$(0 \leqslant \mu \leqslant 1), \quad (0 \leqslant \sigma \leqslant \infty).$$

L'introduction des coordonnées σ et μ elliptiques simplifie notablement toutes les expressions et les calculs.

Posant $\mu = 0$ et laissant σ variable, nous obtenons la partie extérieure $r \geqslant R$ du plan $z = 0$. La partie intérieure $r \leqslant R$ située sous le poinçon correspond à $\sigma = 0$ et $0 \leqslant \mu \leqslant 1$.

Les formules (20) nous fournissent immédiatement les valeurs des dérivées partielles de σ et μ par rapport à z. On trouve en dérivant

$$\left.\begin{aligned} 1 &= R\mu\frac{\partial\sigma}{\partial z} + R\sigma\frac{\partial\mu}{\partial z} \\ 0 &= R\frac{\sqrt{1-\mu^2}}{\sqrt{1+\sigma^2}}\sigma\frac{\partial\sigma}{\partial z} - R\frac{\sqrt{1+\sigma^2}}{\sqrt{1-\mu^2}}\mu\frac{\partial\mu}{\partial z}, \end{aligned}\right\} \tag{21}$$

d'où l'on tire

$$\frac{\partial\sigma}{\partial z} = \frac{\mu(1+\sigma^2)}{R(\mu^2+\sigma^2)}, \quad \frac{\partial\mu}{\partial z} = \frac{\sigma(1-\mu^2)}{R(\mu^2+\sigma^2)}. \tag{22}$$

Introduisant dans l'intégrale (16) les valeurs de Z et B en fonctions de σ et μ et séparant la partie imaginaire de la partie réelle, nous trouvons les valeurs

$$\int_0^\infty e^{-pz} \cos pR J_1(pr)\, dp = \frac{1}{r} \frac{(1-\mu)(\sigma^2-\mu)}{(\mu^2+\sigma^2)} \tag{23}$$

$$\int_0^\infty e^{-pz} \sin pR J_1(pr)\, dp = \frac{1}{r} \frac{\sigma(1-\mu^2)}{(\mu^2+\sigma^2)} = \frac{R}{r} \frac{\partial \mu}{\partial z}. \tag{24}$$

Intégrant par rapport à z les deux membres de la dernière identité (24) on trouvera

$$\int_0^\infty e^{-pz} \frac{\sin pR}{p} J_1(pr)\, dp = -\frac{1}{r} R\mu + C(r). \tag{25}$$

Posant ensuite $\mu = 0$ ce qui correspond à la partie extérieure $r \geqslant R$ du plan $z = 0$ on obtiendra

$$C(r) = \int_0^\infty \frac{\sin pR}{p} J_1(pr)\, dp = \frac{R}{r} \quad (r > R) \tag{26}$$

(on remarquera que cette expression de $C(r)$ reste encore valable pour $r < R$). Ainsi donc nous trouverons la valeur de l'intégrale

$$\int_0^\infty e^{-pz} \frac{\sin pR}{p} J_1(pr)\, dp = \frac{R(1-\mu)}{r} \tag{27}$$

et en définitif la valeur de la dérivée (15)

$$\frac{\partial \varphi_3}{\partial z} = -\frac{4\gamma}{\pi} \cos\vartheta \left[\frac{R(1-\mu)}{r} - R \frac{(1-\mu)(\sigma^2-\mu)}{r(\sigma^2+\mu^2)} \right] =$$
$$= -\frac{4\gamma \cos\vartheta\, R\mu (1-\mu^2)}{\pi r (\sigma^2+\mu^2)} = -\frac{4\gamma r}{(1+\sigma^2)^2} \frac{\partial \sigma}{\partial z} \cos\vartheta. \tag{28}$$

On en déduit immédiatement l'expression pour φ_3 en intégrant

$$\varphi_3 = -\frac{4\gamma r}{2\pi} \left[\frac{\sigma}{1+\sigma^2} + \operatorname{arctg} \sigma \right] \cos\vartheta + C_1(r, \vartheta). \tag{29}$$

Pour avoir la valeur du terme additif $C_1(r, \vartheta)$ d'intégration il suffira de poser $\sigma = 0$ ou bien $z = 0$ et $r < R$. Nous aurons

$$C_1(r, \vartheta) = (\varphi_3)_{z=0, r<R} = \gamma r \cos\vartheta \tag{30}$$

et par suite

$$\varphi_3 = -\frac{\gamma r}{\pi} \left(\frac{2\sigma}{1+\sigma^2} + 2\operatorname{arctg} \sigma - \pi \right) \cos\vartheta. \tag{31}$$

Il nous reste à evaluer la constante γ. Supposant que la valeur du moment M de forces est donnée on aura

$$M = \int Z_z x\, dx\, dy$$

l'intégration étant rapportée sur l'intérieur du cercle $r = R$. De l'expression (31) on tire pour $r < R$ la valeur de Z_z pour $z = 0$

$$(Z_z)_{z=0} = G \frac{m}{m-1} \frac{\partial \varphi_3}{\partial z} = -G \frac{m}{m-1} \cdot \frac{4\gamma R(1-\mu^2)\cos\vartheta}{\pi r \mu} =$$
$$= -G \frac{m}{m-1} \cdot \frac{4\gamma R}{\pi r} \cdot \frac{r^2 \cos\vartheta}{R^2 \sqrt{1 - \frac{r^2}{R^2}}} = -G \frac{m}{m-1} \cdot \frac{4\gamma r \cos\vartheta}{\pi \sqrt{R^2 - r^2}}. \tag{32}$$

On obtiendra donc en intégrant

$$M = \int Z_z\, x\, dx\, dy = \int_0^R \int_0^{2\pi} -\frac{Gm}{m-1} \frac{4\gamma r \cos\gamma}{\pi \sqrt{R^2 - r^2}} r \cos\vartheta\, r dr d\vartheta =$$

$$= -\int_0^R \frac{Gm}{m-1} \frac{4\gamma r^3}{\sqrt{R^2 - r^2}} dr = -\frac{Gm}{m-1} 4\gamma R^2 \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta =$$

$$= -\frac{8}{3} G \frac{m}{m-1} \gamma R^3, \tag{33}$$

et par suite

$$\gamma = -\frac{3}{8} \cdot \frac{m-1}{m} \cdot \frac{M}{GR^3}. \tag{34}$$

Occupons nous à présent des expressions générales pour l'espace de Z_z et w. Nous avons

$$\frac{\partial^2 \varphi_3}{\partial z^2} = \frac{\partial}{\partial \sigma}\left(\frac{\partial \varphi_3}{\partial z}\right) \cdot \frac{\partial \sigma}{\partial z} + \frac{\partial}{\partial \mu}\left(\frac{\partial \varphi_3}{\partial z}\right) \frac{\partial \mu}{\partial z} =$$

$$= \frac{4\gamma \cos\vartheta}{\pi} \cdot \frac{\sigma(1-\mu^2)(\mu^4 + 5\sigma^2\mu^2 + 3\mu^2 - \sigma^2)}{r(\sigma^2 + \mu^2)^3} \tag{35}$$

et par conséquent

$$Z_z = G \frac{m}{m-1}\left(\frac{\partial \varphi_3}{\partial z} - z \frac{\partial^2 \varphi_3}{\partial z^2}\right) =$$

$$= \frac{3M\mu(1-\mu^2)}{2\pi r R^2 (\sigma^2 + \mu^2)^3} (\mu^4 + 5\sigma^2\mu^2 + \mu^4\sigma^2 + 5\sigma^4\mu^2) \cos\vartheta. \tag{36}$$

$$w = \varphi_3 - \frac{m}{2(m-1)} z \frac{\partial \varphi_3}{\partial z} =$$

$$= \frac{3}{4\pi} \frac{M}{GR^3} \left\{ \frac{m-1}{m} r \left(\frac{\sigma}{1+\sigma^2} - \operatorname{arcctg}\sigma \right) - \frac{zR\mu(1-\mu^2)}{r(\sigma^2 + \mu^2)} \right\} \cos\vartheta. \tag{37}$$

2. On peut déduire les formules analogues pour le cas de la distribution symétrique des efforts sous le poinçon.

On trouve les expressions suivantes (2)

$$\varphi_3 = \frac{2k}{\pi} \int_0^\infty \frac{\sin pR}{p} J_0(pr)\, e^{-pz}\, dp = \frac{2k}{\pi} \operatorname{arcctg}\sigma \tag{38}$$

où le paramètre k est lié à la force P de pression donnée par la relation:

$$k = \frac{P(m-1)}{4GmR}. \tag{39}$$

On en déduit en dérivant que

$$\frac{\partial \varphi_3}{\partial z} = \frac{2k}{\pi} \int_0^\infty -\sin pR J_0(pr)\, e^{-pz}\, dp = \frac{\partial \varphi_3}{\partial \sigma} \frac{\partial \sigma}{\partial z} + \frac{\partial \varphi_3}{\partial \mu} \frac{\partial \mu}{\partial z} = \tag{40}$$

$$= -\frac{2k}{\pi} \frac{1}{1+\sigma^2} \frac{\mu(1+\sigma^2)}{R(\mu^2 + \sigma^2)} = -\frac{2k}{\pi R} \frac{\mu}{\mu^2 + \sigma^2},$$

$$\frac{\partial^2 \varphi_3}{\partial z^2} = \frac{2k}{\pi R^2} \frac{\sigma(3\mu^2 - \sigma^2 + 3\mu^2\sigma^2 - \mu^4)}{(\mu^2 + \sigma^2)^3} \tag{41}$$

et par suite [voir aussi (6)]

$$\left.\begin{aligned} Z_z &= G\frac{m}{m-1}\left\{\frac{\partial\varphi_3}{\partial z}-z\frac{\partial^2\varphi_3}{\partial z^2}\right\}= \\ &= -\frac{P}{2\pi R}\left\{\frac{\mu}{R(\mu^2+\sigma^2)}+\frac{z}{R^2}\cdot\frac{\sigma(3\mu^2-\sigma^2+3\mu^2\sigma^2-\mu^2)}{(\mu^2+\sigma^2)^3}\right\}= \\ &= -\frac{P\mu}{2\pi R^2(\mu^2+\sigma^2)^3}(\mu^4+5\mu^2\sigma^2+3\mu^2\sigma^4-\mu^4\sigma^2) \\ w &= \varphi_3-\frac{m}{2(m-1)}z\frac{\partial\varphi_3}{\partial z}=\frac{P}{4\pi RG}\left\{2\frac{m-1}{m}\operatorname{arcctg}\sigma-\frac{z\mu}{R(\sigma^2+\mu^2)}\right\}. \end{aligned}\right\} \quad (42)$$

3. Si l'on suppose qu'une force P est appliquée à un point du poinçon à une distance $x=l$ de son axe, l'action d'une telle force se réduira à celle d'un moment $M=-Pl$ et d'une force P appliquée au centre de gravité. Dans ce cas nous aurons par conséquent

$$Z_z=\frac{P}{2\pi R^2}\left\{\frac{\mu}{(\mu^2+\sigma^2)^3}(\mu^4+5\mu^2\sigma^2+3\mu^2\sigma^4-\mu^4\sigma^2)-\right.$$
$$\left.-\frac{3(1-\mu^2)\mu}{r(\mu^2+\sigma^2)^3}(\mu^4+5\mu^2\sigma^2-2\mu^4\sigma^2+5\sigma^4\mu^2)\,l\cos\vartheta\right\}=$$
$$=-\frac{P\mu}{2\pi R^2(\mu^2+\sigma^2)^3}\left\{\mu^4+5\mu^2\sigma^2+3\mu^2\sigma^4-\mu^4\sigma^2+\right.$$
$$\left.+\frac{3}{r}(1-\mu^2)\,(\mu^4+5\mu^2\sigma^2+\mu^4\sigma^2+5\sigma^4\mu^2)\,l\cos\vartheta\right\}, \quad (43)$$

$$w=\frac{P}{4\pi RG}\left\{2\frac{m-1}{m}\operatorname{arcctg}\sigma+\frac{z\mu}{(\sigma^2+\mu^2)R}-\right.$$
$$\left.-\frac{3}{R^2}\left[\frac{m-1}{m}r\left(\frac{\sigma}{1+\sigma^2}-\operatorname{arcctg}\sigma\right)-\frac{zR\mu(1-\mu^2)}{r(\sigma^2+\mu^2)}\right]l\cos\vartheta\right\}. \quad (44)$$

Sous le poinçon dans la partie intérieure $r\leqslant R$ du plan $z=0$, c'est-à-dire pour $\sigma=0$ nous aurons:

$$\left.\begin{aligned} r=R\sqrt{1-\mu^2},\ Z_z &= \frac{P}{2\pi R^2}\left[-\frac{1}{\mu}-\frac{3}{r}\frac{1-\mu^2}{\mu}\,l\cos\vartheta\right]= \\ &= -\frac{P(3lx+R^2)}{2\pi R^3\sqrt{R^2-r^2}}, \\ w=\frac{m-1}{m}\cdot\frac{P(3lx+2R^2)}{8GR^3},\ \gamma &= \frac{3}{8}\cdot\frac{m-1}{m}\cdot\frac{Pl}{GR^3}. \end{aligned}\right\} \quad (45)$$

L'effort Z_z ne pourra être une pression dans toute l'étendue du cercle $r\leqslant R$ de base du poinçon que sous la condition que l soit inférieur à $\frac{R}{3}$:

$$l\leqslant\frac{R}{3}. \quad (46)$$

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LITTÉRATURE CITÉE

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THEORY OF ELASTICITY

DEFORMATIONS OF NATURALLY TWISTED BARS

By P. RIZ

(Communicated by I. M. Vinogradow, Member of the Academy, 19. IV. 1939)

In this last part of our paper* we consider the flexure of a naturally twisted bar by a couple of the moment M applied at its butt end ($z=0$). We confine ourselves to the case of a uniformly twisted bar $[\alpha(z)=\theta z]$. The equation of its lateral surface is

$$f(x\cos\alpha - y\sin\alpha,\quad x\sin\alpha + y\cos\alpha)=0, \tag{1}$$

or, if we introduce the co-ordinates

$$\xi = x\cos\alpha - y\sin\alpha,\quad \eta = x\sin\alpha + y\cos\alpha,\quad \zeta = z, \tag{2}$$

$$f(\xi,\eta)=0. \tag{3}$$

Let the stresses be

$$\left.\begin{aligned} &\sigma_{zz} = -\beta E x + \beta\theta\sigma_{zz}^{(1)},\quad \sigma_{xx}=\beta\theta\sigma_{xx}^{(1)},\quad \sigma_{yy}=\beta\theta\sigma_{yy}^{(1)},\\ &\sigma_{xy}=\beta\theta\sigma_{xy}^{(1)},\qquad\quad \sigma_{xz}=\beta\theta\sigma_{xz}^{(1)},\quad \sigma_{yz}=\beta\theta\sigma_{yz}^{(1)}, \end{aligned}\right\} \tag{4}$$

where $\beta=\dfrac{M}{EI_\eta}$ and $\sigma_{zz}^{(1)}$ etc. are some unknown functions of ξ, η, ζ. In order to find them up to the terms of the order θ^2 we have to consider the following problem.

The equilibrium equations:

$$L_1^{(1)}=0,\quad L_2^{(1)}=0,\quad L_3^{(1)}=0, \tag{5}$$

where $L_1^{(1)}=\dfrac{\partial\sigma_{xx}^{(1)}}{\partial\xi}+\dfrac{\partial\sigma_{xy}^{(1)}}{\partial\eta}+\dfrac{\partial\sigma_{xz}^{(1)}}{\partial\zeta},\ \ldots$

The conditions on the lateral surface:

$$M_1^{(1)}=0,\quad M_2^{(1)}=0,\quad M_3^{(1)}=E\xi(\xi f'_\eta - \eta f'_\xi), \tag{6}$$

where

$$M_1^{(1)}=\sigma_{xx}^{(1)}f'_\xi+\sigma_{xy}^{(1)}f'_\eta,\ \ldots\ ,\quad M_3^{(1)}=\sigma_{xz}^{(1)}f'_\xi+\sigma_{yz}^{(1)}f'_\eta.$$

* Cf. C. R. Acad. Sci. URSS, XXIII, No. 1, XXIII, No. 5 (1939).

Besides, we have the compatibility conditions and the integral conditions at the butt cross-section expressing the statical equivalence of stresses to the flexing couple. We shall satisfy all the stated conditions, if we put:

$$\left.\begin{aligned}
&\sigma_{xx}^{(1)} = \sigma_{yy}^{(1)} = \sigma_{xy}^{(1)} = 0,\\
&\sigma_{xz}^{(1)} = -E\xi\eta + \frac{\partial\psi}{\partial\xi} + \frac{\partial\psi_0}{\partial\xi} + m\left(\frac{\partial\varphi}{\partial\xi} - \eta\right),\\
&\sigma_{yz}^{(1)} = -\frac{E}{2}\xi^2 + \frac{\partial\psi}{\partial\eta} + \frac{\partial\psi_0}{\partial\eta} + m\left(\frac{\partial\varphi}{\partial\eta} + \xi\right),\\
&\sigma_{zz}^{(1)} = -\frac{E}{I_\xi}(I_\xi - I_\eta)\,\eta\zeta.
\end{aligned}\right\} \tag{7}$$

The function $\psi(\xi, \eta)$ satisfies the equation

$$\nabla^2\psi - E\eta = 0 \tag{8}$$

inside the contour $f(\xi, \eta) = 0$ and the condition

$$\frac{\partial\psi}{\partial n} - \frac{3}{2}E\xi^2 f'_\eta = 0 \tag{9}$$

on the contour. The function $\psi_0(\xi, \eta)$ satisfies the equation

$$\nabla^2\psi_0 + (1+\nu)\frac{I_\xi - I_\eta}{I_\xi}\eta = 0 \tag{10}$$

and the contour condition

$$\frac{\partial\psi_0}{\partial n} = -\frac{\nu E}{I_\xi}(I_\xi - I_\eta)\,\xi^2 f'_\eta, \tag{11}$$

i. e. it represents the flexure function for the contour $f(\xi, \eta) = 0$. The function φ is the torsion function for the same contour. The constant m is determined by the condition that the torque moment should vanish in the butt cross-section.

In spite of the fact that we obtain the additional stresses by multiplication of $\sigma_{xz}^{(1)}$, $\sigma_{yz}^{(1)}$, $\sigma_{zz}^{(1)}$ by a small factor $\beta\theta$, they may, all the same, be in fact quite considerable, since by choosing the cross-section sufficiently small we may make the ratio $\frac{I_\eta}{I_\xi}$ as large as we please.

The displacements u and v do not depend on the form of the functions ψ, ψ_0 and φ and are easily determined in the general case. Having found them, we establish without difficulty the following relations between the curvature of the elastic line and the flexing moment:

$$\frac{dx^2}{dz^2} = -\frac{M}{EI_\eta}, \quad \frac{d^2y}{dz^2} = \frac{M}{E}\theta z\left(\frac{1}{I_\eta} - \frac{1}{I_\xi}\right). \tag{12}$$

These relations coincide up to the terms of the order with the relations deduced by S. Toumarkin (1) from the general equations of Kirchhoff for infinitely thin bars. Observe that on ground of Kirchhoff's theory these relations may be obtained only with the help of additional hypotheses which we do not make here.

The displacement w may be obtained with the help of the functions

ψ, ψ_0 and φ. The function ψ is easily found for those contours, for which the flexure problem is solved, so for instance, for the ellipse:

$$\psi = E\left\{\frac{1}{4}\xi^2\eta - \frac{1}{4}\eta^3 + \frac{1}{3}\,\frac{a^2+2b^2-\frac{3}{4}(b^2-a^2)}{a^2+3b^2}\,(\eta^3 - 3\xi^2\eta) - \frac{b^2(a^2-b^2)}{a^2+3b^2}\,\eta\right\}.$$

The obtained results may be naturally generalized for the case when the flexing moment possesses components acting along the y-axis as well as along the x-axis. The method exposed in the present paper may also serve in the case of the flexure by a force applied at the end, but then the problem is considerably more complicated. Finally, introducing parameters characterizing the shrinking of the bar towards the end, as it has been done in D. Panov's paper [2], or characterizing the bend of its axis, we may consider deformations of naturally twisted bars with varying cross-sections and bent axes. A detailed exposition of these results is given in our paper which is to appear in «Труды ЦАГИ».

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THERMOTECHNICS

THE STATE EQUATION OF REAL GASES

By M. P. VUKALOVICH and I. I. NOVIKOV

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More than a hundred years have passed since the problem was suggested in science to construct the equation of state (characteristic equation) for gases, liquids and solids on the basis of our concepts of molecular structure. The problem has not been solved up to this time, although its practical importance is very great. The aim of the present paper is to find a rational form of characteristic equation for gases by applying methods of chemical equilibrium to phenomena taking place in real gases and to examine the influence of the association and dissociation processes on the parameters characterizing the gaseous state. What is understood here under association of gas molecules is a merely mechanical co-aggregation of two or more molecules into one composite particle acquiring the rôle and the importance of an independent gas particle, but endowed with only some of the properties of a great molecule. In other words, it is supposed that such a co-aggregation is not attended by any positional rearrangement of atoms within the molecule which in associated condition maintains therefore its individual properties. Every gas the molecules of which have co-aggregated, is to be considered therefore as a mixture of a number of gases, the particles of which consist of groups formed by 2, 3, 4 or more simple molecules respectively. To simplify calculation it is assumed that chiefly double molecules, or pairs, are formed on co-aggregation. Then our gas will present a mixture of two gases—of gas I, consisting of single or simple molecules, and of gas II, the particles of which are double molecules. We have before us the task to construct the characteristic equation for such a gas mixture. To each of these gases we can apply any of heretofore suggested equations of state, say, that of Clapeyron which is the simplest of them. The use of this equation implies that each of these gases is considered as ideal one, and the interaction between its particles is neglected. After the equation of state for this ideal mixture has been obtained, the error introduced with the above assumption must be rectified with reference to volume and pressure, as it has been done by van-der-Waals.

Taking one grammolecule of gas we find it thus to contain N_1 single molecules, forming the gas I, and N_2 double molecules, forming the gas II. Besides, $N_1 + 2N_2 = N$, where N is the number of Avogadro, i. e. the number of single molecules in one mole of gas in the absence of association. Applying Clapeyron's law to the gases I and II, we have

for the gas I: $p_1 v = \frac{N_1}{N} RT$ and for the gas II: $p_2 v = \frac{N_2}{N} RT$. Summing up the left and the right members of these equalities respectively, we obtain $(p_1 + p_2) v = \frac{N_1 + N_2}{N} RT$; according to the law of Dalton: $p_1 + p_2 = p$; therefore $pv = \frac{N_1 + N_2}{N} RT$; transforming now the expression $\frac{N_1 + N_2}{N}$ in

$$\frac{N_1 + N_2}{N} = \frac{N_1 + N_2 + N_2 - N_2}{N} = \frac{N_1 + 2N_2 - N_2}{N} = \frac{N - N_2}{N} = 1 - \frac{N_2}{N},$$

we get $pv = \frac{N_1 + N_2}{N} RT = R\left(1 - \frac{N_2}{N}\right) T$ or taking $R\left(1 - \frac{N_2}{N}\right) = R'$, we have $pv = R\left(1 - \frac{N_2}{N}\right) T = R'T$.

The equation of state for the ideal associating gas is drawn up. It has the form $pv = R'T$ and differs from Clapeyron's equation in that the value R' is smaller than R and variable, depending obviously on the state parameters. Let us endeavour to find out this dependence. With that purpose in mind we will more attentively examine the processes taking place in every associating gas, in which, as we assumed formerly, only double molecules are formed and which therefore represent a mixture of two «gases» I and II. Between these gases a peculiar interaction is observed: the double molecules disaggregate into single ones, while the single molecules co-aggregate into pairs. If the system is in equilibrium, the decrease of double molecules is exactly counterbalanced by the formation of new pairs of single molecules. The process is fully analogous to a chemical reaction of the type: $I + I \rightleftarrows II$, where the molecules of gas I combine to form molecules of gas II, while the molecules of the latter break down into molecules of the former. If the general thermodynamic law characterizing the course of the chemical reactions of a similar type, i. e. the law of mass action, is resorted to, we get

$$\prod K_i^{\nu_i} = cp^{-\Sigma\nu_i} T^{\frac{\Sigma\nu_i c_{pi}}{k}} e^{-\frac{\Sigma\nu_i U_{0i}}{kT}},$$

where K_i is the concentration of the i-th partner of the reaction; c_{p_i} is the specific heat at $p = \text{const}$ per one molecule; U_{o_i} is the equilibrium energy of the atoms in one gas molecule or the heat of its formation; $k = \frac{R}{N}$ is a Boltzmann constant; c is a constant; ν_i is the number of gram-moles of the i-th substance participating in the reaction. It has a negative sign for the initial products of reaction and a positive sign for the final products.

In the process which we have represented symbolically by the reactions $I + I \rightleftarrows II$, two moles of gas I associate to form one mole of gas II and vice versa; therefore, $\nu_1 = -2$ and $\nu_2 = 1$; then the law of mass action assumes the form

$$K_1^{-2} K_2 = cp^{-(-2+1)} T^{\frac{-2c_{p_1} + c_{p_2}}{k}} e^{-\frac{-2U_{01} + U_{02}}{kT}}$$

or

$$\frac{K_2}{K_1^2} = cpT^{\frac{-2c_{p_1} + c_{p_2}}{k}} e^{-\frac{2U_{01} + U_{02}}{kT}} \qquad (1)$$

Besides (1), another equation emerges from the definition of concentration:

$$K_1 + K_2 = 1, \qquad (2)$$

i. e. the sum of concentrations of all the substances, participating in the reaction, is equal to unity. From equations (1) and (2) we can find degrees of concentration K_1 and K_2, if we know the values c_{p_2} and U_{02}, i. e. the heat capacity and the equilibrium energy of double molecules which are included in the expression for the law of mass action. We have therefore to determine c_{p_2} and U_{02} and for doing this will inquire into the mechanism of formation of double molecules. We suppose that the main factor in the formation of pairs is van-der-Waals' interaction between single molecules, and that it is just by the energy of this interaction that two single molecules are linked in a pair. Under ordinary conditions when the temperature is not too low, the energy of van-der-Waals' interaction is known to be smaller than kT, hence it is sufficiently small (kT is here the average kinetic energy assigned to one degree of freedom). From the fact that the value of the linkage energy is so small, follows the conclusion that the formation of a double molecule from two simple ones is effected without rearrangement of their atoms and that consequently a pair of associated molecules can be considered a mere mechanical aggregate of particles which have preserved their individual properties. Another conclusion to be drawn from the weakness of the union between two molecules forming a pair, is that the rotation of such a double molecule as a whole about any of three possible axes cannot possibly produce any change in its heat capacity, for the energy of this rotation is of the same order of magnitude as the union energy, i.e. considerably less than kT; therefore the energy of such rotations will be neglected in all our further calculations.

As mentioned above, the ordinary molecules aggregated into a double one maintain every kind of their movements, although they fail to realize them completely, i. e. not all the movements possible in a simple molecule are realized, when it has become part of a pair.

Let us examine the way in which the associated molecule effects its rotation about its own centre of gravity. Suppose the double molecule to be in such a state when the rotations in the composing molecules are not yet excited and can be provoked only in the future through a collision with other molecules. But the energy developed on collision is of the same order of magnitude as is kT, i. e. it considerably exceeds the co-aggregation energy and the blow thus experienced by the pair is extremely dangerous for its existence and in most of the cases will result in disaggregation, while such collision as would excite the simple molecules to rotation without destroying the pair is a matter of very great luck. Therefore we are entitled to suppose that in a double molecule the rotations of the composing single molecules are likely to be absent partly or completely, i. e. not all of the possible rotations are realized. Part of them is lost and the proportion of those non-effected rotations depends on the structure and symmetry of the molecules, being obviously the higher, the more symmetrical is their structure. The number of the lost degrees of rotation in a single molecule denoted by m can either be equal to zero or assume whole values—1, 2, 3...

Thus, for monoatomic gases where the rotation of molecules is in general absent, $m=0$; for diatomic, triatomic and polyatomic rod-like molecules (within which the atoms are arranged in alignement, say, in the case of CO_2), we can expect that $m=1$; for the most symmetrical molecules of the type CH_4 (say, benzene and other substances) $m=3$; for the molecules of an intermediate type and particularly for water vapour $m=2$. It should be noted, however, that a more precise manner is to assign to the number m not whole but fractional values near to and always smaller than 1, 2, 3. The reason of this is that the non-effected rotations of simple molecules are not enti-

rely lost and partly exert their influence. In the same direction are acting the rotation of the double molecule as a whole and the vibrations of its component molecules with respect to each other; both these factors have been neglected by us; consequently, the number of rotations affected by the simple molecules of a pair is equal to $2\,(3-m)=6-2\,m$ rather than 6 (three rotations per molecule). As to the vibratory movements of atoms within a simple molecule, we assume them completely maintained. The assumptions made will suffice for calculating the specific heat c_{p_2} of a double molecule and the equilibrium energy U_{02} of its atoms. Indeed, in calculating the specific heat we must take into consideration that in comparison with two single molecules a double one possesses $3+2\,m$ degrees of freedom less (3 degrees at the expense of the progressive movement and 2 m rotations lost by component molecules).

Hence, according to the kinetic theory the specific heat of a pair will be by $\frac{3+2m}{2}k$ less than the twofold specific heat of a single molecule, i. e. $c_{v_2}=2c_{v_1}-\frac{3+2m}{2}k$. Taking into consideration the relation $c_p=c_v+k$ for one molecule we find that

$$c_{p_2}=c_{v_2}+k=2c_{v_1}-\frac{3+2m}{2}k+k$$

or substituting c_{v_1} with c_{p_1}

$$c_{p_2}=2(c_{p_1}-k)-\frac{3+2m}{2}k+k=2c_{p_1}-\frac{3+2m}{2}k-k=2c_{p_1}-\frac{5+2m}{2}k.$$

As to the equilibrium energy U_{02}, we find in accordance with the structural scheme of our double molecule that $U_{02}=2U_{01}+U_{012}$, while U_{012} is the equilibrium energy of van-der-Waals' interaction between co-aggregated molecules, and U_{01} is the eqilibrium energy of atoms in a simple molecule. Substituting the determined values for c_{p_2} and U_{02} in the expression of the law of mass action, we get

$$\frac{K_2}{K_1^2}=cpT^{\frac{-2c_{p_1}+2c_{p_1}-\frac{5+2m}{2}k}{k}}e^{-\frac{-2U_{01}+2U_{01}-U_{012}}{kT}}.$$

After reduction while making good use of the relation $U_{012}\ll kT$, we obtain definitely $\frac{K_2}{K_1^2}=cpT^{-\frac{5+2m}{2}}$. Replace K_1 and K_2 by $K_1=\frac{N_1}{N_1+N_2}$ and $K_2=\frac{N_2}{N_1+N_2}$ respectively and the equation will take the form

$$\frac{N_2}{N_1^2}(N_1+N_2)=cpT^{-\frac{5+2m}{2}}.$$

The equation of state for a mixture of the ideal «gases» I and II has the form

$$pv=\frac{N_1+N_2}{N}RT,\quad \text{wherefrom}\quad p=\frac{N_1+N_2}{N}RT\frac{1}{v}.$$

If this expression stands for p in the equation $\frac{N_2}{N_1^2}(N_1+N_2)=cpT^{-\frac{5+2m}{2}}$, the latter assumes the following form

$$\frac{N_2}{N_1^2}=\frac{cR}{N}\frac{1}{T^{\frac{3+2m}{2}}}\frac{1}{v}.$$

Denote $cRT^{-\frac{3+2m}{2}}\frac{1}{v}=L$.

Resolving the equations obtained for N_1 and N_2 on the assumption that α is small, we find by expansion in series

$$N_1 = N(1-2\alpha+8\alpha^2\ldots) \quad \text{and} \quad N_2 = N(\alpha-4\alpha^2\ldots).$$

Replacing N_1 and N_2 in the equation $pv = \frac{N_1+N_2}{N} RT$ by their expression we obtain:

$$pv = RT(1-\alpha+4\alpha^2\ldots)$$

where $\alpha = cRT^{-\frac{3+2m}{2}} \frac{1}{v}$.

The equation obtained is an equation of state for an ideal associating gas, i. e. such a gas in which the effect of molecular interaction is only to call forth co-aggregation of gas molecules, and does not lead to corrections for pressure and volume. In order to take account of the latter we have to suppose that in the first approximation we can apply to each of our gases the law of van-der-Waals. Then we may by way of analogy write down the equation for a real associating gas in the following manner:

$$\left(p+\frac{a}{v^2}\right)(v-b) = \frac{N_1+N_2}{N} RT \quad \text{or} \quad pv = \frac{\frac{N_1+N_2}{N} RT}{1-\frac{b}{v}} - \frac{a}{v};$$

substituting $\frac{1}{1-\frac{b}{v}}$ for $1+\frac{b}{v}$, we obtain

$$pv = \frac{N_1+N_2}{N} RT\left(1-\frac{b}{v}\right) - \frac{a}{v},$$

since

$$N_1+N_2 = N(1-\alpha+4\alpha^2\ldots)$$

or

$$N_1+N_2 = N\left(1-f_1(T)\frac{1}{v}-f_2(T)\frac{1}{v^2}\ldots\right).$$

We obtain the equation of state for real gases in the following ultimate form:

$$pv = RT\left[1-A(T)\frac{1}{v}-B(T)\frac{1}{v^2}\ldots\right]$$

where

$$A = \frac{a}{RT}+\frac{cR}{T^{\frac{3+2m}{2}}}-b; \quad B = \frac{bcR}{T^{\frac{3+2m}{2}}}.$$

The calculations performed bring into evidence that the consideration of double molecules gives already a good coincidence with observation. Thus, e. g. for water vapour with pressures up to 225 atm. and temperatures ranging from saturation point up to 550°C the maximum departure (near saturation, pressure over 100 atm.) does not exceed 4%. For diatomic gases and air it does not surpass 1—2%. If a more precise equation of state is endeavoured, larger associations than pairs must also be taken into consideration. Thus, considering triple molecules besides double ones and performing the transformations in much the same manner we see that the equation of state for real gases assumes the former aspect:

$$pv = RT\left[1-A(T)\frac{1}{v}-B(T)\frac{1}{v^2}\ldots\right]$$

Here the coefficients A and B have the following meaning:

$$A = \frac{a}{RT} + \frac{cR}{T^{\frac{3+2m_1}{2}}} - b; \quad B = \frac{bcR}{T^{\frac{3+2m_1}{2}}} + 4\left(1 - \frac{k}{T^{\frac{3m_2-4m_1}{2}}}\right) \cdot \left(\frac{cR}{T^{\frac{3+2m_1}{2}}} \frac{1}{\varphi}\right)^2,$$

where m_2 is equal to m_1 or exceeds it. The calculations for water vapour done by means of this equation yield values of volume coinciding throughout the range of pressures and temperatures with the steam tables of Knoblauch, Mollier, Koch and also with the latest figures agreed internationally. In the case of diatomic gases and air the precision of the data calculated is achieved by considering formation of double molecules only.

It is well to emphasize that the authors of the present paper have deliberately resorted to most simple regularities and particularly to the law of mass action, deduced from the simplest equation of state for ideal gases. But it was equally possible to reach our object by way of a more rigorous and accurate demonstration. However, the exactness which is always desirable from the theoretical point of view, would have changed nothing in the principles of the method proposed by the authors, being only apt to encumber the calculation and make it little accessible to every day practical work. Besides, it has been mentioned already that the use of more simple regularities in our demonstration has provided such accuracy (up to the fifth decimal sign) that meets the requirements of most careful technical calculation. Reference may be here made also to the work of M. Born, K. Fuchs, Meier and others «Statistical mechanics of condensing systems» published in the Proceedings of the Royal Society, June 1938. Starting from the same premises those authors have made a much complicated and difficult mathematical way indicated by statistical mechanics. The equation they have arrived at has the form of an infinite series of volume powers:

$$p\varphi = RT\left[1 - \sum_{\nu=1}^{\infty} \frac{\nu}{\nu+1} \frac{\beta\nu}{v^\nu}\right],$$

where the coefficients $\beta\nu$ are some indefinite functions of the temperature T, represented by the so-called integrals of power functions. But it is the form of our equation too with the only difference that in our equation the coefficients A and B represent the well-known temperature functions, whereas in the equations of Born they are indefinite. The association theory put forward by us has made it possible also to solve question as to the specific heat of real gases. This work will be reported in our next articles.

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PHYSICS

ON QUANTUM ELECTRODYNAMICS. I

By V. L. GINSBURG

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The Hamilton operator for the non-relativistic electron* and field has, as is well known, the following form:

$$H=\frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}(r)\right)^2+e\varphi(r)+\frac{1}{8\pi}\int[(\vec{E}^{tr})^2+\vec{H}^2]dv, \qquad (1)$$

where $\vec{A}(r)$ is the transverse vector potential at the spot where the electron lies. With this

$$\operatorname{div}\vec{A}=0,\ \vec{E}^{tr}=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t};\ \vec{H}=\operatorname{rot}\vec{A}.$$

Usually, upon the assumption that no external magnetic field is available, all the terms of (1) comprising $\vec{A}$ are treated as the energy of light quanta or the energy of interaction between the electron and the light quanta.

Such an interpretation fails, however, to stand in conformity with classical electrodynamics, as it should be anticipated. In fact, a field of moving charges even in the absence of any radiation cannot be described only by a longitudinal vector and scalar potentials, since, in particular, the magnetic field accompanying a moving charge refers wholly to the transverse vectorial potential. Thus, in the classical theory a transverse field contains both the light part and the part carried along with the charges which is radically different from light. The same situation takes place in quantum electrodynamics, too, in so far as the division of the vector potential into a longitudinal and a transverse part is relativistically non-invariant, and should, for instance, the energy of the transverse field in one system of co-ordinates be equal to zero, in the other system it will fail to be so equal. Now, we know that a light field cannot be created by the selection of a system of co-ordinates. We are, therefore, obliged to recognize as erroneous the opinion, according to which in the quantum theory the light field is entirely a field of light.

The advanced considerations permit to comprehend the nature of the emission of an electromagnetic field by a uniformly moving electron which appeared specific for the quantum electrodynamics.

V. A. Fock (1) has emphasized the fact that the wave equation of quantum electrodynamics with Hamilton's operator (1) or a corresponding relativistic one fails to have a stationary solution corresponding in the absence of light quanta in the field to a uniformly moving electron, whence it follows that a uniformly moving electron must be endowed with radiation.

* We confine our attention from the very beginning to the non-relativistic case, solely with a view to making our further account more conveniently understood.

A. Smirnov (2) has calculated this radiation in the non-relativistic case making use of the wave functions of the 1st approximation of Shrödinger's theory of disturbance and neglecting the radiation reaction upon the electron. He obtained the following results.

With the adiabatic inclusion of the interaction of the electron with the field, the electron emits an energy equalling:

$$H_{tr} = \frac{4\pi e^2}{m^2 c^2} \sum_{\lambda} \frac{(\vec{p}\,\vec{e}_\lambda)^2}{\vec{k}_\lambda^{\,2}}. \qquad (2)$$

In the case of instantaneously included interaction we calculate the mean (with respect to time) value of emitted energy which is equal to the energy of the field with $t \to \infty$. In this case we have

$$\overline{H}_{tr}^{(\text{mean as to } t)} = \underset{t\to\infty}{H_{tr}} = \frac{8\pi e^2}{m^2 c^2} \sum_{\lambda} \frac{(\vec{p}\,\vec{e}_\lambda)}{\vec{k}_\lambda^{\,2}}. \qquad (3)$$

In formulae (2) and (3) $\vec{p}$ is the impulse of the electron, $\vec{e}_\lambda$ and $\vec{k}_\lambda$ — the unit vector of polarization and the wave vector of photons of λ kind. In the respective formulae of Smirnov we have passed over to ordinary units from Heaviside units; we have assumed Hohlraum's volume to be equal to unity and have replaced summation over all directions of $\vec{k}_\lambda$ by summation over a half of the directions of the wave vector.

The emission of electromagnetic energy by a uniformly moving electron mentioned above is accounted for by the kind of the initial conditions selected. In fact, the initial conditions resorted to in quantum electrodynamics and according to which there exists with $t=0$ a uniformly moving electron, while the energy of the transverse field is equal to zero, signify that when $t=0$ the moving charge is not surrounded by entrained magnetic and transverse electric fields. Such initial conditions cannot possibly correspond to stationary state and necessarily lead to the fact that, upon the inclusion of interaction, the electron will emit at least its own entrained field.

The corresponding problem might be stated in terms of the classical theory, and all Smirnov's results would thereby be obtained. This is precisely performed hereafter, while some other questions pertaining to the one under discussion will be considered in another communication.

Hamilton's function in classical electrodynamics for a non-relativistic electron and field has form (1), where the available quantities will, of course, no longer be considered as operators. Expanding $\vec{A}$ into flat waves:

$$\vec{A}(\vec{x}, t) = \sum_{\lambda} (q_{\lambda_1} \vec{A}_{\lambda_1} + q_{\lambda_2} \vec{A}_{\lambda_2}), \qquad (4)$$

where $\vec{A}_{\lambda_1} = \vec{e}_\lambda \sqrt{8\pi}\, c \cos(\vec{k}_\lambda \vec{x})$ and $\vec{A}_{\lambda_2} = \vec{e}_\lambda \sqrt{8\pi}\, c \sin(\vec{k}_\lambda \vec{x})$, we obtain for H_{tr} the expression:

$$H_{tr} = \frac{1}{8\pi} \int [(\vec{E}^{tr})^2 + \vec{H}^2]\, dv = \frac{1}{2} \sum_{\lambda, i} (p_{\lambda i}^2 + \nu_\lambda^2 q_{\lambda i}^2), \qquad (5)$$

where $p_{\lambda i} = \dot{q}_{\lambda i}$, $\nu_\lambda = c\,|\vec{k}_\lambda|$ and $i = 1.2$. In (4) and (5) the summation is effected over one half of the directions $\vec{k}_\lambda$. The co-ordinates of field $q_{\lambda i}$ are subordinate to the following system of equations:

$$\ddot{q}_{\lambda i} + \nu_\lambda^2 q_{\lambda i} = \frac{e}{cm} (\vec{p} - \frac{e}{c} \vec{A}(r),\ \vec{A}_{\lambda i}(r)). \qquad (6)$$

Function (1) and the equations of motion following therefrom take due account of both the influence of the field upon the particles and, conversely, of the particles upon the field. The solution of this general problem appears to be outside the limits of possibility. Consequently, inasmuch as we are primarily concerned with the influence of particles upon field, i. e. emission, we shall make assumptions which are equivalent to neglecting the reaction of radiation upon the electron. Neglecting the reaction signifies that we neglect the emission impulse and this is attained by assuming that in (1) $\vec{k}_\lambda = 0$. In this case in the absence of external fields $[\varphi(r) = 0]$, as will be easily seen, the impulse of particle $\vec{p}$ is retained.

Besides this, let us neglect term $\frac{e^2}{2mc^2}\vec{A}^2(r)$ in (1) upon which, as can be seen from (5), depends the connexion between the oscillators of the field.

The assumptions made (which are equivalent to those of Smirnov) lead to the following form of the Hamilton function:

$$H = \frac{1}{2m}\vec{p}^2 - \frac{e\sqrt{8\pi}}{m}\sum_\lambda (\vec{p}\vec{l}_\lambda^2)\, q_{\lambda_1} + \frac{1}{2}\sum_{\lambda, i}(p_{\lambda i}^2 + \nu_\lambda q_{\lambda i}). \tag{7}$$

Equations (6) assume the form:

$$\ddot{q}_{\lambda_1} + \nu_\lambda^2 q_{\lambda_1} = \frac{e\sqrt{8\pi}}{m}(\vec{p}\vec{e}_\lambda),\ \ddot{q}_{\lambda_2} + \nu_\lambda^2 q_{\lambda_2} = 0. \tag{8}$$

Their solution allowing for $\vec{p}$=const is as follows:

$$q_{\lambda 1} = \frac{e\sqrt{8\pi}}{m}\frac{(\vec{p}\vec{e}_\lambda)}{\nu_\lambda^2} + q'_{\lambda 1},\ q_{\lambda 2} = q'_{\lambda 2}, \tag{9}$$

where $q'_{\lambda i}$ are solutions of a system of homogeneous equations corresponding to system (8).

If at the initial moment there exists a uniformly moving electron unsurrounded by a transverse field, and if we adiabatically include the interaction between the electron and the field, then in so far as adiabatic inclusion corresponds physically to a infinitely slow acceleration of the electron up to state of motion with impulse $\vec{p}$, the electron will emit only its own entrained field. The energy of the latter will be obtained provided a stationary solution of equations (8) is substituted into (5), since the stationary solution obtained in the case of $q'_{\lambda i} = 0$ corresponds precisely to the availability of a single entrained field. Here we properly come to expression (2) which, upon the introduction of density of conditions and integration over angles, becomes transformed into

$$H_{tr} = \frac{4e^2}{3\pi c^3}\left(\frac{\vec{p}^2}{2m^2}\right)\int_0^{\nu_{max}} d\nu, \tag{10}$$

where ν_{max} is the frequency at which we break off the spectrum. In the case of a point electron, maximum frequency fails to exist and, as should be expected, electromagnetic mass $M = \frac{4e^2}{3\pi c^3}\int_0^{\nu_{max}} d\nu$ is infinite.

The instantaneous inclusion of interaction corresponds to the instantaneous imparting of impulse $\vec{p}$ to the electron. Therefore now for the determination of H_{tr} the solutions (8) satisfying the initial conditions:

$$q_{\lambda i} = \dot{q}_{\lambda i} = 0 \qquad \text{when } t = 0 \tag{11}$$

must be substituted into (5).

As a result we have:

$$H_{tr} = \frac{8\pi e^2}{m^2 c^2} \sum_{\lambda} \frac{\vec{p}\,\vec{e}_\lambda}{\vec{k}_\lambda^2} \{1 - \cos(c|\vec{k}_\lambda| t)\}. \tag{12}$$

If the spectrum be broken off at frequency ν_{max}, we obtain:

$$H_{tr} = \frac{8e^2}{3\pi c^3} \left(\frac{\vec{p}^2}{2m}\right) \left(\nu_{max} - \frac{\sin \nu_{max} t}{t}\right). \tag{12'}$$

It will be easily seen that $\underset{t\to\infty}{H_{tr}} = \overline{H}^{(\text{mean as to } t, \text{ where } t \text{ is great})}$ is exactly equal to expression (3).

The emitted energy is greater with the instantaneous inclusion of the disturbance than it is with adiabatic inclusion, since with instantaneous inclusion there is an acceleration at the initial moment and the electron also emits a light field besides the entrained field.

In case if the electron possesses a magnetic moment and if at the outset the energy of the transverse field is equal to zero, then, with the advent of interaction, magnetic energy corresponding to this magnetic moment must necessarily also be radiated, and indeed, it may be readily shown by supplementing (1) by term $-\vec{\mu}\vec{H}$ (where $\vec{\mu}$ is the magnetic moment) that the electron upon adiabatic inclusion of interaction emits, besides the above-mentioned energy, also that of the magnetic moment, this being true both for the quantum and the classical case:

$$H_{tr} = \frac{e^2\hbar^2}{2\pi m^2 c^5} \int_0^{\nu_{max}} \nu^2 \, d\nu. \tag{13}$$

The same assumptions for the initial equations are here made as before. Besides this, in the classical case it is assumed that $\vec{\mu}^2 = \frac{3}{4} \frac{\hbar^2 c^2}{m^2 c^2}$. In the quantum case we made use of Pauli's equation.

The entrained field may in a number of cases be separated from the light field and this on the basis of the following considerations: the general solution of homogeneous equations corresponding to system (6) always describes a light field, since through its substitution into (4) flat transverse waves moving with velocity c will be available. Conversely, a particular solution of equation (6)* gives, generally speaking, no light waves at all. Thus, in the case of a uniformly moving electron, if we replace in (6) $\frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}(r)\right)$ by $\vec{v} = \text{const}$, and if we do not assume $\vec{k}_\lambda = 0$, we obtain a particular solution which, with substitution into (4), gives a simple expansion of the vectorial potential of the entrained field into a Fourier series, i. e. it describes only the entrained field. In the general case, the existence of frequency regions will be possible in which the solution of homogeneous and heterogeneous equations describes physically undistinguishable fields. As such regions we have resonance regions with the availability of which the electron is liable to radiate.

Continuing our examination of a uniformly moving electron let us proceed in (7) having the outward appearance in classical as well as in quantum theo-

* If the canonical form of equations be abandoned, expression $\frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}(r)\right)$ should be replaced by velocity of electron $\vec{v}$ which, if we neglect the reaction, does not depend upon the co-ordinates of the field.

ries* to new variables $q'_{\lambda i}$ determinable from (9). Variables $q'_{\lambda i}$ subject to homogeneous equations describe the light field. Therefore the accomplished transformation whose supplement up to the canonical is trivial ($\vec{p}' = p$; $p'_{\lambda i} = p_{\lambda i}$) has the sense of a distinction between the entrained and the light field. In the new variables (7) assumes the following form:

$$H = \frac{1}{2m}\vec{p}'^2 - \frac{8\pi e^2}{m^2c^2}\sum_{\lambda}\frac{(\vec{p}'\vec{e}_\lambda)}{\vec{k}_\lambda^2} + \frac{4\pi e^2}{m^2c^2}\sum_{\lambda}\frac{(\vec{p}'\vec{e}_\lambda)}{\vec{k}_\lambda^2} + \frac{1}{2}\sum_{\lambda, i}(p'^2_{\lambda i} + \nu_\lambda^2 q'^2_{\lambda i}), \qquad (14)$$

where, of course, in the quantum case $\vec{p}'$, $p'_{\lambda i}$ and $q'_{\lambda i}$ are operators.

The separation of variables obtained in (14) is natural, since on the assumptions made regarding the absence of reaction and dispersion, the uniformly moving electron is not liable to interact with the light field. The second term in (14) representing the energy of interaction of the electron with its own field diverges and together with its similar third term [in (14)] gives in the quantum region the so-called transverse divergence. The third term in (14) has become segregated from H_{tr} in (7) and is precisely the energy of the entrained field.

Let us remark that an operator of the type of (14) was obtained by Pauli and Fierz [3] in connexion with a consideration of other questions.

Results wholly analogous to the above-stated are obtainable if the magnetic electron is considered with the help of Pauli's equation in the quantum region and its corresponding equation in the classical region. In the case of Dirac's electron, it will be possible to obtain a separation of variables and a segregation of the entrained field by making assumptions which are equivalent to renouncing to take any account of the reaction of radiation upon the electron, as pointed out by Bloch and Nordsieck [4]. Expression (14) shows also that in considered case the entrained field does not quantate. More exactly speaking, its quantization, as well as the quantization of the longitudinal field is left out of consideration on account of the connexion between the separate oscillators of the field caused by the availability of particles (hence the heterogeneity of the equations). It will be important to emphasize the fact that while this omission of quantization always holds true for a longitudinal entrained field, for a transverse field such a statement evidently ceases to be correct.

In point of fact, in regions where the entrained field is undistinguishable from the light one (see above) its quantization should be substantial.

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* Equations (6) and (8) remain in force. too, in the quantum theory if the following notations are adopted: $\dot{q}_{\lambda i} = \frac{1}{ih}\{q_{\lambda i}H - Hq_{\lambda i}\}$. Expressions (9) remain to be solutions of equations (8), if H has the form of (7).

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PHYSICS

IONIZATION OF THE ATOM DUE TO BETA-DECAY

By E. L. FEINBERG

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I. A β-particle, when formed in the course of β-decay, may: a) radiate a γ-quantum when losing its energy in the field of its own nucleus ([1,2]); b) annihilate with the core electron in the case of positron decay ([3]); c) create a pair in the field of its own nucleus ([4,5]), and, finally, d) supply the core electron with the energy; this results in the excitation or ionization of the core of the radioactive atom. Processes a) and b) involve an interaction with the field. Therefore, their probabilities are proportional to $\alpha\sim(137)^{-1}$. In cases c) and d) interaction between the particles takes place and their respective probabilities are of the order of $W\sim\alpha^2$. However, in the case of annihilation b) W is proportional also to the density of the core electrons in the place where the nucleus is situated. This gives an extra-factor $(\alpha Z)^3$ which is very small in all cases of β^+-decay. For the process c) the calculation of W gives a numerical factor $\sim 2\cdot 10^{-3}$. The probability of both effects is therefore much lower than that given by the usual evaluation quoted above. We want to show that the ionization d) is, on the contrary, m u c h m o r e p r o b a b l e because W turns out to be inversely proportional to the binding energy of the core electron $I\sim(\alpha Z)^2$. Hence here $W\sim(\alpha Z)^{-2}\alpha^2\sim Z^{-2}$. The external effect of the process will consist in the ejection of slow electrons and the emission of the radiation after the transition of the core into its normal state. It is thus essential in the experiments on detection of neutrino, of the type of those carried out by Crane and Halpern ([6]). As has been pointed out by Wertenstein ([7]), these experiments have been incorrectly interpreted by their authors.

To investigate the processes mentioned above one can use either: 1) the method of spherical waves ([1,3,4]) when the perturbation of the wave (it describes the β-particle) propagating from the nucleus by interaction with other particles (or the field) is considered, or 2) the method conventionally called here the Fermi method, which includes in the calculations the process of the β-decay itself ([1,2,5]). The improvement ([4,5]) which makes it possible to take into account the effect of the extra-charge left in the nucleus when the β-particle is emerging, plays a great rôle. Therefore the surrounding suffers a double action; first, from the β-particle, and secondly, from the charge created in the nucleus. Thus, we have to deal with a problem in which not only the field, but the number of particles as well is changed.

II. To get a clear idea of the importance of individual effects, and master the mathematical difficulties which are especially increased by the necessity of using the exact functions of the continuous spectrum for hydrogen

atom, we shall take advantage of some peculiarities of the problem. Let us consider the „autoexcitation" in the case of a negative electron (for sake of concretisation) β-decay, using the Fermi method. The Hamiltonian of the whole system has the form:

$$H = H_0^{(z)} + H_{02}^{(z)} + H_s + GU + \frac{1-\rho}{2}\left(-\frac{e^2}{r_1} - \frac{e^2}{r_2} + V_{12} + H_{01}^{(z)} + H_\sigma\right).$$

The operators describe: $H_0^{(z)}$ — the unchanging part of the nucleus with the charge Ze; $H_{02}^{(z)}$ — the excited electron when in the field of this constant part (for the case of pair creation it is the electron of the negative spectrum); we shall supply this electron with the index «2»; H_s — the decaying heavy particle; GU — its interaction with the field of light particles; H_σ — the escaped neutrino; $H_{01}^{(z)}$ — the β-electron when in the field of decaying nucleus Z (we supply it with the index «1»); V_{12} — the interaction of the electrons with one another; $-\frac{e^2}{r_1}$, $-\frac{e^2}{r_2}$ — their interaction with the proton created by the decay; ρ is the inner co-ordinate of the heavy particle. The latter operators are present only if the decay has already taken place, $\rho = -1$. In the initial state we have the nucleus Z (we do not write its function), the neutron and the electron «2»: $\Psi^0 = \psi_2^0(\vec{r}_2)\,\psi_s^0(\vec{r}_s)\,\delta_{\rho,+1}$. The Hamiltonian H may be split into the unperturbed part and the perturbation in different ways, for instance:

$$H = H_A^0 + H_A';\quad H_A' = GU + \left(V_{12} - \frac{e^2}{r_2}\right)\frac{1-\rho}{2}; \tag{A}$$

$$H = H_B^0 + H_B';\quad H_B' = GU + V_{12}\,\frac{1-\rho}{2}; \tag{B}$$

$$H = H_C^0 + H_C';\quad H_C' = GU + \left(-\frac{e^2}{r_1} - \frac{e^2}{r_2}\right)\frac{1-\rho}{2}. \tag{C}$$

This is only possible in virtue of the situation that the initial state belongs simultaneously to three different systems of functions of all the unperturbed Hamiltonians:

$$\Psi_A = \begin{pmatrix} \psi_2^{(z)}\psi_s\,\delta_{\rho,+1} \\ \psi_2^{(z)}\psi_1^{(z+1)}\psi_\sigma\psi_s\delta_{\rho,-1} \end{pmatrix};\quad \Psi_B = \begin{pmatrix} \psi_2^{(z)}\psi_s\,\delta_{\rho,+1} \\ \psi_2^{(z+1)}\psi_1^{(z+1)}\psi_\sigma\psi_s\delta_{\rho,-1} \end{pmatrix}$$

$$\Psi_C = \begin{pmatrix} \psi_2^{(z)}\psi_s\,\delta_{\rho,+1} \\ \psi_2^{(z)}(\vec{r}_1,\vec{r}_2)\,\psi_\sigma\psi_s\delta_{\rho,-1} \end{pmatrix} \tag{1}$$

Here, for example, $\psi_2^{(z+1)}$ denotes that the electron function is written for the field of the nucleus $Z+1$; $\psi^{(z)}(\vec{r}_1,\vec{r}_2)$ is the function of both interacting electrons in the field of the nucleus Z in the case [see (C)] their variables are not separable.

In the case (A) for the transition into the final state $\Psi_A^f \sim \psi_2^f\psi_1^f\psi_s^f\psi_\sigma\delta_{\rho,-1}$ in the first approximation of the perturbation theory the matrix element $M_A^{\mathrm{I}} = (f\,|\,H_A'\,|\,0) \equiv 0$. In the second approximation

$$M_A^{\mathrm{II}} = \sum_i \frac{1}{E_0 - E_i}(2^f 1^f\,|\,V_{12}\,|\,2^0 1^i)(2^0 1^i\,\sigma\rho^f\,|\,GU\,|\,2^0\rho^0) + \\ + \frac{1}{E_2^0 - E_2}\left(2^f\left|\frac{e^2}{r_2}\right|2^0\right)(1^f\sigma\rho^f\,|\,GU\,|\,\rho^0). \tag{2}$$

The first term here describes the «direct collision» (d. c.). The meaning of the second one becomes clearer when using the method (B) in which

case already the first order transition is possible in virtue of the non-orthogonality of functions $\psi_2^{(z)}$ and $\psi_2^{(z+1)}$:

$$M_B^{\mathrm{I}} = (2^{f(z+1)}\, 1^f\, \sigma\rho^f \,|\, GU \,|\, 2^{0\,(z)}\, \rho^0) \neq 0. \tag{3}$$

It is easy to see that it corresponds to the second term in (2) and describes the change in the state of the core, due to the change in the nucleus charge. Therefore, this change of Z results not only in the «adiabatic» shrinkage of the core, but in the excitations of different kind as well. This appears as the «shaking» of the core *. Besides, d.c. is acting, too.

$$M_B^{\mathrm{II}} = \sum_{ik} \frac{1}{E_0 - E_{ik}} (2^f\, 1^f \,|\, V_{12} \,|\, 2^i\, 1^k)\, (2^i\, 1^k\, \sigma\rho^f \,|\, GU \,|\, 2^0\, \rho^0),\ M_B = M_B^{\mathrm{I}} + M_B^{\mathrm{II}}. \tag{4}$$

This method is in principle more accurate than (A), because it includes more terms of H in the zero order approximation. On the other hand, the calculation of the d. c. here is not so simple. Thus, we have to use both methods. However, for slow β-particles when the functions become more complicated it is hardly possible to calculate the d. c. in the second approximation. At the same time, the method of spherical waves is not permitted by fundamental considerations: for slow β-particles we can no longer say that we operate in the «wave zone» in the electrodynamical sense. In the latter case we are interested in the method (C).

For the transition into the state $\Psi^f = \psi^{(z)\,f}(\vec{r_1}\,\vec{r_2})\, \psi_s \psi_s^f \delta_{\rho,-1}$ here the first order transition is possible in virtue of non-orthogonality of functions $\psi^{(z)\,f}(\vec{r_1}\,\vec{r_2})$ and $\psi_1^0 \psi_2^0$. It is due wholly to the transfer of V_{12} into the unperturbed Hamiltonian H_C^0. Thus, it expresses the effect of d. c.

$$M_C^{\mathrm{I}} = ((1,2)^f\, \sigma\rho^f \,|\, GU \,|\, 2^0 \rho^0). \tag{5}$$

The main mathematical difficulty consists in the calculation of d. c. and in proving that it is unessential in comparison with the shaking. The simplest method of calculation of the matrix element for d. c. is that of spherical waves, which is permitted for fast β-particles.

The utilization in different combinations according to particular conditions of all these four methods comprises the essential feature of the present work.

For the calculations, the non-relativistic approximation was adopted which proved quite sufficient**.

III. The fast β-particles (the effect of nuclear charge on the β-particle functions may be neglected). It may be shown very generally that for any transition of the second electron the method of sphe-

* It follows that the creation of a pair investigated in ([4,5]) which is due mostly to the effect of increasing nuclear charge, results essentially from the non-orthogonality of the electron functions of the negative energy spectrum in the field of the nucleus Z and those of the positive energy spectrum in the field $Z+1$. The product of such functions is markedly different from zero, only if their absolute energy is large enough. However, in the latter case they are reduced nearly completely to the plane waves and are nearly orthogonal. This explains why the effect is so small. Indeed, having written for this case each function as the sum of a plane wave and a small addition term, due to perturbation Ze^2/r or $(Z+1)e^2/r$, and having constructed the non-orthogonality integral, we get formula (8) in ([5]). The only difference is in the absence of the factor which takes into account the retardation. In our Hamiltonian the field is not included because in the present case retardation is not essential.

** The relativistic count made thereupon has confirmed the results obtained and shown that the knocking-out of K-electrons at high energy states ($E_2 \gg mc^2$) is hardly probable, except for very heavy elements, where $W \sim 10^{-5}$, too.

rical waves and those of Fermi in the form (A) which was used by Arley and Möller [5] give identical results. Thus, calculation may be effected using any of them. After this, the more accurate method (B) was applied. We calculated the probabilities of ionization of the K-shell.

a) For the beginning, the electron «2» in the final state was described by plane waves (the Born approximation). The calculation by the method of spherical waves has shown that the d. c. is less important than shaking, their radiation being I_K/E (I_K—ionization potential of K-shell, E—the energy of β-particle). Therefore, it may be omitted for light elements. This being done, W_K is independent of E, while the energy of emerged electrons E_2 is small ($\sim I_K$). This result justifies the utilization of non-relativistic approximation, but prohibits the Born approximation for the electron «2».

b) Accordingly, the calculations were repeated with the exact functions of the continuous spectrum. When calculating the matrix element of V_{12} the additional simplification was admitted: $E_2=0$. The transition into the states with $l=0$ and 1 were investigated. In this case, too, d. c. proves to be less important than shaking in the same proportion I_K/E.

c) Having proved that d. c. is unessential and accordingly omitted the perturbation V_{12}, we were able to calculate W_K by the most accurate method (B). Here the difference between the cases β^+, β^--decay appeared. We obtained the following number of electrons knocked-out of the K-shell per 1 β-decay

$$W_K=\frac{64}{3}\left(1+\frac{2}{3}\varepsilon\right)\varepsilon e^{-4\varepsilon}\frac{1}{Z^2};\quad \varepsilon=\frac{Z'}{Z}=\frac{Z\pm 1}{Z}, \qquad (6)$$

where $Z'e$ stands for nuclear charge after the β-decay.

For $\varepsilon\to 1$, $W_K\to\sim 0.64Z^{-2}$, this coincides with the results obtained by method (A). Calculating the transition to the levels with $E_2<0$ we receive in one particular case the probability (it is great, of course) of adiabatic shrinkage of the shell. Thus, for small Z (light atoms, probably also the outer electrons of heavy atoms) W_K is much larger than $\alpha^2\sim 10^{-4}$.

IV. Slow β-particles. As was mentioned above, the effect of d. c. increases with decrease of E. However, for small E the Born approximation for β-particle is not allowed and additional investigation is necessary. This becomes possible by using method (C). To obtain a rough idea of the effect, the excitation of K electron on the L_{II} level, not the ionization, was calculated. It turned out that even here the excitation due to d. c. is much less than that due to shaking. If this result is valid for all types of excitation, we may conclude that the d. c. effect may be altogether neglected (this conclusion can not, however, be regarded as an ultimate one). On the other hand, concussion was calculated precisely using the method (B) and proved to be independent of E. For slow β-particles we should only make allowance for the effect of exchange. This somewhat reduces the result. A full account will be published elsewhere.

I take this opportunity for expressing my thanks to Prof. Ig. Tamm who has suggested the theme and given many valuable advices.

A more detailed account will be published elsewhere.

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PROPAGATION OF WAVES IN A LIQUID POSSESSING MAXWELL VISCOSITY

By M. A. ISSAKOVICH

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In his treatise «On the dynamical theory of gases» (1) Maxwell considers, inter alia, an impermeable medium for which internal shearing stress S is connected with shear-deformation e by the equation:

$$\dot{S} = a\dot{e} - \frac{S}{\tau}. \qquad (1)$$

Here a and τ are physical constants of the medium, a being the modulus of elasticity and τ—relaxation time. A dot on top of the respective letter stands for substantial differentiation with respect to time.

If the motion of the medium is such that the tensor of velocities of deformation fails to remain constant for every particle, then the arising stresses will not be in linear dependence upon deformation rates and the behaviour of the medium itself will depend upon the degree of the non-stationarity of motion. We shall call a similar medium «a relaxating liquid», or a liquid possessing Maxwell viscosity.

Let us consider non-stationarity great if the changes in stress during relaxation were great compared to stress magnitude, and consider it small in the opposite case. In the first instance, if we neglect quantity $\frac{S}{\tau}$ as compared with $\dot{S}$ in equation (1), we shall arrive at Hook's law for an elastic body with modulus of elasticity a. In the second case by neglecting magnitude $\dot{S}$ in comparison with $\frac{S}{\tau}$ we arrive at the equation of liquid friction. Thus, for a relaxating liquid the notion of Newton viscosity is generally inapplicable; with sufficiently small non-stationarity the behaviour of such a liquid coincides practically with the behaviour of an ordinary viscous liquid with a viscosity coefficient of $\mu = \frac{1}{2} a\tau$; in the terminal case of great non-stationarity, the stresses will conform to Hook's law. Substances of the type of pitch or sealing-wax behave at least qualitatively in accordance with equation (1).

Certain questions referring to the hydrodynamic aspect of the problem of a relaxating liquid have been discussed by the writer in the paper: „On the hydrodynamics of a liquid possessing Maxwell viscosity".

Special interest belongs to the problem of the propagation of sound waves in a relaxating liquid. Of late much attention has also been given to the propagation of ultra-sound in both gases and liquids; indeed, quite a number

of experimental works have been devoted to this problem. Certain phenomena observed in course of the propagation of ultra-sound seem not to be interpretable in terms of Navier-Stokes' equation, even if we are to assume that the second viscosity coefficient differs from zero. It may be supposed that the phenomenon of viscosity is inherently a relaxation phenomenon corresponding to Maxwell's scheme. Such a supposition with reference to the second viscosity coefficient (voluminar relaxation) was in fact put forward by L. I. Mandelstam and M. A. Leontovich in their work: „A contribution to the theory of the absorption of sound in liquids" (2) where the respective calculations are also made. Introducing a similar presumption with respect to the first viscosity coefficient, it is possible to assume that ordinary viscosity too is explained by the relaxation of shearing stresses and that, which is especially the case for highly viscous liquids, the effect of relaxation begins to influence the course of the phenomenon appreciably with ultra-acoustic frequencies. In accordance with this, acting on the proposal of L. I. Mandelstam, we are here endeavouring to consider the problem of the propagation of compression waves in a liquid possessing not only the voluminar, but the shearing relaxation as well.

Let us recall in this connexion that the propagation of sound vibrations in a medium possessing relaxation properties was already investigated theoretically by A. Gemant (3). However, both the basic prerequisites of his theory and the results of his investigation differ radically from what is given below.

Let us generalize Maxwell's equation (1) for the case of any deformation of a liquid particle, presupposing the availability of both shear and voluminar relaxation.

Let us assume that the condition of the liquid is any stated moment fully determined by its density ρ and the difference $e_{ik}-\xi_{ik}$ of two tensors e_{ik} and ξ_{ik} (i, $k=1, 2, 3$). We shall determine tensor e_{ik} as a tensor satisfying, in a Cartesian system of co-ordinates, to the equation

$$\dot{e}_{ik}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right) \tag{2}$$

where u_i is the component of velocity over axis x_i. Thus, $\dot{e}_{ik}$ is the tensor of deformation velocities, while for small deformations (e. g. for small vibrations) e_{ik} is the tensor of deformation. Let us determine tensor ξ_{ik} as a tensor corresponding to the so-called equation of reaction:

$$\dot{\xi}_{ik}=\frac{1}{\tau_1}(e_{ik}-\xi_{ik})+\delta_{ik}\,\frac{1}{3}\left(\frac{1}{\tau_2}-\frac{1}{\tau_1}\right)(e_{aa}-\xi_{aa}). \tag{3}$$

Here τ_1 is the time of relaxation of shearing stresses, while τ_2 is time of voluminar relaxation.

The components of tensor ξ_{ik} characterize in a certain sense the internal condition of the liquid. We do not ascribe to them any mechanical import that could be visualized. As it seems, they were introduced for the first time into hydrodynamic equations by L. Natanson in his paper „On the laws of viscosity" (4).

In the mentioned paper on voluminar relaxation the same parameters have been used. Assuming the linearity of the dependence between tensor of stresses in liquid S_{ik} and tensor $e_{ik}-\xi_{ik}$ we shall obtain for the general case of the availability of voluminar relaxation and shear relaxation (owing to the symmetricity of S_{ik}) the following expression:

$$S_{ik}=-\delta_{ik}p+a(e_{ik}-\xi_{ik})+\delta_{ik}\frac{1}{3}(\varkappa-a)(e_{aa}-\xi_{aa}). \tag{4}$$

Here p is hydrostatic pressure, a is modulus of shear elasticity, $\varkappa$—modulus of voluminar elasticity (for the problems we are interested in, $\varkappa$ is adiabatic modulus). Quantities

$$\mu = \frac{1}{2} a\tau_1$$

and

$$\mu' = \frac{1}{3} \varkappa\tau_2 \quad \left(\nu = \frac{1}{2}\frac{a\tau_1}{\rho} \quad \text{and} \quad \nu' = \frac{1}{3}\frac{\varkappa\tau_2}{\rho}\right)$$

correspond to the first and second dynamic (kinematic) coefficients of viscosity. Excluding for the case of pure shear parameter ξ_{ik} from equations (3) and (4), let us return to Maxwell's equation (1a).

Let mass force F_i act upon the liquid; then, considering the forces acting upon an element of the liquid and taking (4) into account, we find:

$$\rho\dot{u}_i = \rho F_i - \frac{\partial p}{\partial x_i} + a\frac{\partial}{\partial x_k}(e_{ik} - \xi_{ik}) + \frac{1}{3}(\varkappa - a)\frac{\partial}{\partial x_i}(e_{\alpha\alpha} - \xi_{\alpha\alpha}). \qquad (5)$$

Equations (5), (2) and (3) are a system of hydrodynamic equations of motion for a liquid endowed with Maxwell viscosity.

R. Reiger [5] has considered the propagation of transverse vibrations in a relaxating liquid. The results of his investigation can be obtained from the above cited hydrodynamic equations. It is necessary to emphasize the important circumstance that for sufficiently great frequencies, in a liquid possessing Maxwell viscosity it is quite possible to observe typical poorly damping transverse vibrations standing at variance with the case of usual viscosity.

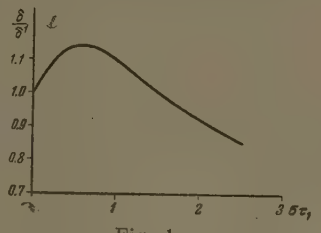

Fig. 1.

If transverse vibrations of liquid are caused by the vibrations of a plate in its own plane caused by the action of elastic forces (the plate in question coming in contact with the surface of the liquid), then, as will be easily derived from the basic equations, damping coefficient δ for the vibrations of a plate is equal to:

$$\delta = \frac{\rho}{\rho_0}\sqrt{\frac{\sigma\nu}{2}}\,\frac{\sqrt{\sigma\tau_1 + \sqrt{\sigma^2\tau_1^2+1}}}{\sqrt{\sigma^2\tau_1^2+1}},$$

where ρ_0 is surface density of plate and σ is cyclic frequency of vibrations. The respective magnitude of damping δ' in the absence of relaxation is equal to $\delta' = \frac{\rho}{\rho_0}\sqrt{\frac{\sigma\nu}{2}}$. The dependence of the ratio $\frac{\delta}{\delta'}$ upon the frequency of vibrations is given in the graph.

Let us now consider, confining ourselves to the case of small vibrations, how a flat sound wave of cyclic frequency σ is propagated in the direction of axis x_1. The velocities of the particles of the liquid are in this case directed parallel to axis x_1 and the equation for the motion of the liquid assumes the appearance:

$$\rho\frac{\partial u_1}{\partial t} = -\frac{\partial p}{\partial x_1} + a\frac{\partial}{\partial x_1}(e_{11} - \xi_{11}) + \frac{1}{3}(\varkappa - a)\frac{\partial}{\partial x_1}(e_{\alpha\alpha} - \xi_{\alpha\alpha}). \qquad (6)$$

Denoting by c Laplacian velocity of sound with small acoustic frequencies and by s — the ratio of the increment of density to the density of the undisturbed liquid, we shall obtain, as usual, $p = p_0 + c^2\rho s$, where p_0 is pressure in a non-disturbed liquid. Compression s satisfies equation of continuity:

$$\frac{\partial s}{\partial t} = -\frac{\partial u_1}{\partial x_1}. \qquad (7)$$

Excluding the magnitudes ξ, p, s from the equations of motion, of continuity and of reaction we, noting that $\dot{e}_{11}=\frac{\partial u_1}{\partial x_1}$, shall obtain the equation of transverse vibrations of the liquid in the form:

$$\tau_1\tau_2\frac{\partial^4 u_1}{\partial t^4}-\tau_1\tau_2\left(c^2+\frac{2}{3}\frac{a}{\rho}+\frac{1}{3}\frac{\varkappa}{\rho}\right)\frac{\partial^4 u_1}{\partial t^2\partial x_1^2}+$$
$$+(\tau_1\tau_2)\frac{\partial^2 u_1}{\partial t^2}-\left[c^2\ (\tau_1+\tau_2)+\frac{2}{3}\frac{a\tau_1}{\rho}+\frac{1}{3}\frac{\varkappa\tau_2}{\rho}\right]\frac{\partial^3 u_1}{\partial t\partial x_1^2}-$$
$$-c^2\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial t^2}=0. \qquad (8)$$

Substituting into equation (8) periodical solution

$$u_1=e^{i\sigma t-mx}$$

we obtain the complex wave number m in the form

$$m=i\frac{\sigma}{c}\left(1+\frac{2}{3}\frac{a}{c^2\rho}\frac{\sigma^2\tau_1^2}{1+\sigma^2\tau_1^2}+\frac{1}{3}\frac{\varkappa}{c^2\rho}\frac{\sigma^2\tau_2^2}{1+\sigma^2\tau_2^2}+\right.$$
$$\left.+i\frac{2}{3}\frac{a}{c^2\rho}\frac{\sigma\tau_1}{1+\sigma^2\tau_1^2}+i\frac{1}{3}\frac{\varkappa}{c^2\rho}\frac{\sigma\tau_2}{1+\sigma^2\tau_2^2}\right)^{-\frac{1}{2}}.$$

Assuming dispersion and damping to be small, we obtain:

$$m=i\frac{\sigma}{c_\sigma}+\alpha=i\frac{\sigma}{c\left(1+\frac{2}{3}\frac{\nu}{c^2}\frac{\sigma^2\tau_1}{1+\sigma^2}+\frac{1}{2}\frac{\nu'}{c^3}\frac{\sigma^2\tau_2}{1+\sigma^2\tau_2^2}+\right.}+$$
$$\left.+\frac{2}{3}\frac{\nu}{c^3}\frac{\sigma}{1+\sigma^2\tau_1^2}+\frac{1}{2}\frac{\nu'}{c^3}\frac{\sigma^2}{1+\sigma^2\tau_1^2}\right.$$

The velocity of propagation of waves of frequency σ is equal to

$$c_\sigma=c\left(1+\frac{2}{3}\frac{\nu}{c^2}\frac{\sigma^2\tau_1}{1+\sigma^2\tau_1^2}+\frac{1}{2}\frac{\nu'}{c^2}\frac{\sigma^2\tau_2}{1+\sigma^2\tau_2^2}\right).$$

The coefficient of damping is equal to

$$\alpha=\frac{2}{3}\frac{\nu}{c^3}\frac{\sigma^2}{1+\sigma^2\tau_1^2}+\frac{1}{2}\frac{\nu'}{c^3}\frac{\sigma^2}{1+\sigma^2\tau_2^2}.$$

From the expression for velocity it follows that in a liquid possessing Maxwell viscosity „relaxation" dispersion takes place.

With $\sigma\tau_1\gg|(\sigma\tau_2\gg 1|)$ change in velocity and coefficient of damping, determined by shear relaxation (voluminar relaxation), assume terminal values (the phenomenon of „saturation"), while damping along one and the same wave length tends to become zero with increased frequency. With small frequencies, we obtain the ordinary coefficient of damping for a viscous liquid with two coefficients of viscosity ν and ν'. The terminal values of damping caused by shear relaxation and voluminar relaxation are respectively equal to $\frac{1}{3}\frac{a}{\tau_1 c^3\rho}$ and $\frac{1}{6}\frac{\varkappa}{\tau_2 c^3\rho}$. The interesting point to note here is that if we are to decrease τ_1 or τ_2 leaving a and $\varkappa$ without change, viscosity coefficients μ and μ' will decline, while the terminal values for damping will increase. Let us note further that in contradistinction to the results obtained the value of the damping coefficient for great frequencies obtained from A. Gemant's

theory of complex viscosity is proportional to the square or frequency, just as is the case for usual viscosity.

It is possible to expect that for highly viscous liquids the influence of the second coefficient of viscosity with ultra-acoustic frequencies is relatively small, while dispersion and damping caused by shear relaxation are already apt to differ from the respective magnitudes for an ordinary viscous liquid. It is possible that this very phenomenon took place in Zachoval's experiments, who observed an increase in the velocity of sound in castor oil; according to this writer, velocity in passing from a frequency of $1.25 \cdot 10^7$ to a frequency of $3.75 \cdot 10^{-7}$ increased by a magnitude of the order of 1%. In this case with a given frequency and fixed ν, the maximum possible value for the velocity of propagation in a relaxating liquid corresponds to relaxation time $\tau_1 = \frac{1}{\nu}$ and is equal to $c\left(1+\frac{1}{3}\frac{c\nu}{c^2}\right)$. Let us note in conclusion that Maxwell's simple scheme cannot claim to give a quantitative explanation of all the phenomena involved in the propagation of ultra-sound. It is, however, possible to assume that it is fully capable of correctly accounting for quite a number of caracteristic phenomena.

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PHYSICAL CHEMISTRY

ON THE CRYSTALS OF QUARTZ GLASS

By N. A. SHISHAKOV

(Communicated by Wl. A. Kistiakowsky, Member of the Academy, 10. IV. 1939)

Among modern theories of vitreous state that of Zachariasen (1) has so far met with the greatest success. According to this theory, glasses are formed of a three-dimensional net-work with no symmetry and periodicity, yet resembling a monocrystal in other respects. Zachariasen's theory won a particularly high recognition after Warren's X-ray investigations (2), who photometered his X-ray diagrams and compared the obtained data with those intensities which had been calculated theoretically on the basis of different hypotheses regarding atomic distribution in glass. In this way the distance between various atoms were found, such as those of Si, O, Na and others. This theory received, indeed, such a widespread acceptance that even the authors of very serious investigations (3) were afterwards compelled to renounce their contentions (4).

We need to mention this success of Zachariasen-Warren's theory only in order to show to what extent the theoretical conceptions of other investigators were inconsistent. Yet it will be hardly possible to agree with the theory itself, and this on the basis of the following considerations.

The fact that with the help of the method of guessings it is possible to obtain curves of intensity distribution which coincide more or less with experimental curves is not yet sufficient proof of the advanced hypotheses regarding atomic distribution. Similar curves may also be obtained with other presumptions. Moreover, notwithstanding the possibility of choosing among various suppositions, the desired coincidence of theoretical and experimental curves was not always obtained. While Warren succeeded in obtaining for fused silica a good coincidence of the first strong maximum observed with the calculated one, nevertheless in the case of glass made of germanium oxide, the agreement was obtained only in the position of the maximum, while its form appeared to be different in the theory and in the experiment. The second Warren's methods of radial distribution curve yields no exact results for great distances between atoms.

Zachariasen-Warren's theory accounts for but very few of the properties of glass. Many of these properties and in particular those which are of the greatest practical significance, for example, mechanical, electrical and chemical ones, fall practically out of the scope of this theory. There are good reasons to believe that in all further attempts to account for these properties Zachariasen's theory would encounter quite a number of difficulties.

Table 1

Results of X-Ray (X) and Electronographical (E) Investigations of Quartz Glass SiO_2

Clark 1929 X	Randall 1930 X	Warren 1933 X	Lewin and Ott 1933 X	Dixit 1935 E	Maxwell and Moseley 1935 E	Shishakov 1935 E	Hartleif (ionization method) 1938			
							1st sample	2nd sample	3rd sample	4th sample
$d=7.15$	—	—	—	—	—	—	very str. 8.33	very str. 8.33	very strong broad 6.25	very strong broad 6.25
	—	—	—	—	—	—	—	—		
—	4.33	4.32	4.27	—	strong broad 4.2	weak 4.40 very str. 4.03	very str. 4.03	strong 4.10		
—	—	—	—	—	strong 3.7	strong 3.64 very strong 2.52	weak, very broad 2.7	very broad 2.7	very broad weak 3.1	strong broad 2.8
2.52	—	—	—	—	broad 2.52	—	—	—	—	—
—	—	—	—	—	—	weak 2.17	weak 1.9	weak 1.8	—	—
—	—	—	—	—	—	» 1.99				
—	—	—	—	—	—	» 1.65				
—	1.5	—	1.61	1.55	—	medium 1.47	strong 1.47	strong 1.47	strong 1.47	strong 1.47
—	—	—	—	—	—	weak 1.42	—	—	—	—
—	—	—	1.24	—	strong broad 1.25	medium 1.26	strong 1.20	strong 1.20	strong 1.20	strong 1.09
—	—	—	—	—	—	weak 1.11				
—	—	—	1.12	—	—	» 1.01				

Zachariasen-Warren's conception that the glasses are amorphous stand in contradiction with the results which were obtained by the author ([5]) in 1935 by means of the electron diffraction method. In spite of the predictions of the theory in question, quartz glass yields several sharp lines on the electronograms (compare Table 1). That no doubts can be placed upon those results, is evidenced by the failure of all attempts to adduce sound arguments against them as well as by a number of facts brought to light since then which, on the contrary, plead in their favour.

The usual objection against the interpretation of the electronographical experiments of the author is based upon the apparent discrepancy between X-ray diagrams and electronograms and may, in the last account, be formulated as a contention to the effect that the bright and sharply marked lines on our electronograms might have resulted from the presence of certain impurities in the specimen.

At present it appears quite possible to answer, first, that there exist no contradictions between X-ray diagrams and electronograms and, secondly, to throw some new light on the nature of the crystals of which quartz glass consists. In Table 1 is to be found comparison of the reflexions $\frac{\lambda}{2 \sin \vartheta}$ of X-rays and electron-rays from quartz glass observed by several authors ([2],[3],[5],[6],[7]). In the light of this evidence, as well as on the ground of all the data found in the literature it will be possible to draw the following conclusion: a sufficient number of sharp reflexions have been observed only by the author of the present paper. The absence in electronograms of a reflexion which would correspond to a periodicity of 7—8Å may be explained by its falling into the region of the strong background which always arises in the vicinity of the central spot wherefore it often remains unobserved. The diagrams obtained by other authors contain only a few diffuse reflexions. Their diffuse outline forbids to decide whether single or two and more superimposed reflexions have been recorded. However, comparison of all these data with one another permits to affirm that all these diffuse reflexions lie practically at those places where, according to electronograms, the most intensive reflexions are likely to be found. Weaker reflexions merge in the deep background and with the modern technique of X-ray investigations are not detectable.

As regards the nature of the crystals of quartz glass, the question has not yet been definitely cleared, but available evidence ([5],[8]) permits to maintain that the attempts of explanations formerly ([8]) undertaken are generally in the right direction. In order to illustrate the results obtained along these lines, Table 2 is given, where for convenience of comparison electronographic data for quartz glass are tabulated next to the latest, precise, also electronographic data of the author for two-dimensional or scaly crystals, whose sole or at least principal components are sheets of Si_2O_5. For greater ease of comparison all electronographic data taken from Table 1 are magnified by $1^1/_2\%$. This magnification, though not altogether compulsory, is quite admissible, since the wave lengths of electrons were calculated in those experiments by means of sphere gap, under which conditions the error might have even exceeded these $1^1/_2\%$.

These data are of course insufficient to provide an exhaustive answer to the problem of the structure of quartz glass, but nevertheless the table affords in both cases the possibility to ascertain the existence of lines characteristic for two-dimensional crystals of Si_2O_5. The fact that there are some lines in excess may possibly be due to the presence of third dimension in quartz glass crystals. This question was partly discussed previously ([8]).

Thus, the crystalline structure of quartz glass can at present hardly lead to any doubts and we may hold it as definitely established that whatever

Table 2

Electronograms Taken from Quartz Glass and Tchassov Yarski Clay

Quartz glass (1935)		Tchassov-Yarski clay (1938) *		
Intensity	$\frac{\lambda}{2 \sin \vartheta}$	Intensity	$\frac{\lambda}{2 \sin \vartheta}$	Indices
weak	4.48	very strong	4.470	(100)
very strong	4.11	—	—	—
strong	3.71	—	—	—
very strong	2.57	very strong	2.581	(110)
weak	2.21	weak	2.235	(200)
weak	1.68	medium	1.689	(210)
medium	1.50	strong	1.490	(300)
weak	1.45	—	—	—
medium	1.29	medium	1.290	(220)
very weak	1.13	weak	1.120	(400)
very weak	1.03	weak	1.028	(320)

these crystals may be as a whole, the presence of two-dimensional layers of Si_2O_5 in them should be considered as a trait just as peculiar for their structure as this is recognized in the case of the crystals of mica, clay, pumice stone etc. For this reason it appears altogether impossible to accept Zachariasen-Warren's theory either with regard to SiO_2 quartz glass, or more particularly for many-component glasses, in respect of which it would appear to be particularly risky to draw any conclusion from the poor X-ray pictures.

In closing this article, it might be worth mentioning that Zachariasen-Warren's theory stands, evidently, in contradiction with Hartleif's [7] works, published in 1938. Instead of determining the intensity of diffracted X-rays by means of a microphotometer as other authors had done, this author had recourse to a direct ionization method which appeared to be considerably more precise. In course of his experiments Hartleif succeeded in detecting in a number of his specimens the presence of a sharp maximum corresponding to a periodicity of 8.33 Å, which can hardly be explained by Zachariasen-Warren's theory. Let us by the way remark that a maximum very close to the aforesaid had been detected earlier by Clark and co-workers [3] (cf. Table 1) which fact is not to be ignored at present.

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* Detailed results of the investigation of the clays are to be published in the near future.

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PHYSICAL CHEMISTRY

ON THE MOSAIC BLOCKS OF CRYSTALLINE QUARTZ

By N. A. SHISHAKOV

(Communicated by Wl. A. Kistiakowsky, Member of the Academy, 1. IV. 1939)

In contradiction to the widely spread opinion which is apt to regard silicate glasses as continuous substances similar to liquids, the writer has lately endeavoured (1) to put forward the claim that these glasses have a mosaic structure. The following contention lies at the back of the opinion that in the case of quartz glass the computed density of the crystals (2) occurring in this glass is found to be approximately 5% greater than the measured density of the glass itself. The theoretical idea emerging from this fact permits to advance an explanation for a number of properties of glass which, on the one hand, have a practical importance and, on the other, have not been accounted for by previous theories. The object of the present work is to expound the view that the advancement of conclusions of such a kind, i. e. resting no considerations of density, is sufficiently legitimate.

Crystalline quartz belongs to the class of substances which have been sufficiently well investigated. Its density is 2.649 (3). The parameters of its lattice are: $a=4.89$ Å, $c=5.38$ Å. Knowing, however, the volume of the crystalline cell

$$v=\frac{\sqrt{3}}{2}a^2c,$$

the number of molecules in it $N=3$ and assuming molecular weight $M=60.06$ and the mass of the hydrogen atom $m_H=1.662\cdot10^{-24}$ (4), we may calculate this density by the equation

$$\rho=\frac{2NMm_H}{\sqrt{3}\,a^2c}.$$

Calculation gives $\rho=2.688$, i. e. a quantity exceeding by 1.5% density as measured. Therefore, in the case of crystalline quartz, we are left with the belief that it fails to be an ideal crystal, but contains some kind of cavities.

As in the case of silicate glasses, the character of these cavities is brought out by the phenomenon of the peptization of its powder by water. Under the action of water, the finest particle (obtained by grinding) break off from the quartz grains forming a very slowly precipitating suspension. As is also the case with glasses, this thin suspension will, upon standing, break up into clearly visible striations (table), this phenomenon being here too explained by the discontinuous character of the distribution of particles in respect of their magnitude. The calculation of their equivalent radii r according to Stokes' formula permits of ascertaining a certain regularity in this distribution which is illustrated in columns 3,4 and 5 of the table. Therefore, we are

led to believe that the pores bear the character of regularly distributed cracks, i. e. that quartz possesses a mosaic structure.

Kahlbaum Quartz Kept under Water for about Two Months (position of bands was recorded 24 days after sedimentation set in).

h, cm	r, Å	Δr, Å	n	$\Delta r/n$
3.55	672			
3.95	708	36	2	18
4.24	734	26	1	26
4.50	756	22	1	22
4.72	774	18	1	18
5.00	797	23	1	23
5.30	821	24	1	24
5.71	852	31	2	16
6.04	876	24	1	24
6.28	893	17	1	17
6.50	909	16	1	16
7.04	946	37	2	18
7.40	970	24	1	24
8.12	1015	45	2	22
8.68	1050	35	2	18
8.95	1066	16	1	16
9.30	1087	21	1	21
9.62	1105	18	1	18
10.02	1127	22	1	22
10.38	1148	21	1	21
10.75	1168	20	1	20
11.06	1185	17	1	17
11.36	1201	16	1	16
		Mean	—	20 Å

In the present case only the smallest particles are of interest, namely those for which the upper boundary of the suspension is shifted 3.5 cm in 24 days. The magnitude λ of such a particle is here 1344 Å. In another experiment we obtain $\lambda = 1228$ Å. Therefore the mean magnitude is to be held as lying at 1300 Å. From this magnitude and from the general volume of pores equal to 1.5%, it will be possible as in the case of quartz glass, to calculate approximately the thickness of the cracks. We have

$$d = \frac{\varrho}{3V}\lambda = \frac{0.015}{3} \cdot 1300 = 6.5\text{Å}.$$

Such a magnitude is fit to account for several different properties of quartz crystals, e. g. its ability to retain a great many regularly distributed inclusions (as in amethysts), the penetration of cations through the mineral upon the application of potential difference and several other properties.

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CHEMISTRY

CONTACT CONVERSION OF THE SIX-MEMBERED RING INTO THE FIVE-MEMBERED ONE

By N. D. ZELINSKY, Member of the Academy, and J. A. ARBUZOV

The impellent idea directing our former as well as our current investigations is the statement made many years ago by Mendeleyeff and confirmed by theoretical considerations, according to which a simple contact may be often sufficient to cause the one or the other course of a chemical process.

In every catalytic action we naturally have to do with a contact, but this does not mean that in a number of purely contact reactions it is necessary to confine oneself to the classical conception of catalysis, when the presence of small quantities of a catalyst or ferment increases the velocity of the process and gives it a considerable quantitative expression. The experimental material reported briefly in this paper is a sufficient proof that in this case we are dealing with a contact process and not with catalysis.

Inoue ([1]), for instance, in his paper on the action of the Japanese acid earth on cyclohexanol considers this earth as a catalyst. According to our opinion, this does not agree with facts. In general in the literature we may very often find investigations where purely contact reactions are considered as catalytic ones.

We were interested in the problem of the probable behaviour of the unsaturated or saturated six-membered cycle when brought into contact with metal oxides at such high temperatures at which these cycles undergo no changes unless brought into contact with oxides.

Aluminium Oxide and Cyclohexene. Cyclohexene (110 g) freshly distilled over metallic sodium (b. p. 83—84°/775 mm; $n_D^{15}=1.4480$) was brought into contact with aluminium oxide at a temperature of 450° at a rate of 5.5—6 g of the hydrocarbon per hour. The aluminium oxide was of a commercial grade and contained some silica. The inner diameter of the reaction tube was 16 mm. The layer of aluminium oxide had a length of 60 cm. A condensate (105.4 g) was obtained which was subjected to fractional distillation over sodium. The following fractions were obtained:

1. 75—80°	64.8 g	$n_D^{18}=1.4381$
2. 80—84°	33.6 »	» $=1.4420$

Both fractions were united and again passed over the contact under the same conditions. Through fractional distillation the main part (87.5 g) boiling at 73—84° was collected. This portion was passed over the contact for

the third time and the condensate was subjected to fractional distillation. The following fractions were collected:

1. 71—73°	1.3 g	$n_D^{20}=1.4299$
2. 73—77°	53.6 »	» $=1.4322$
3. 77—80°	14.7 »	» $=1.4359$
4. 80—84°	6.7 »	» $=1.4390$

In this experiment only 9 g of higher boiling products remained in the distillation flask. The greater part of these products passed over in the interval of 175—245° and had a refractive index $n_D^{20}=1.506$.

Fraction 73—77° was unsaturated. In order to determine its nature this fraction (34.6 g) was hydrogenated over platinized charcoal at 150°. The condensate (32.4 g) had a saturated character and had $n_D^{19}=1.4141$. To determine whether this product of hydrogenation contained some cyclohexane, it was dehydrogenated over the same platinized charcoal at 300° with the same rate of contact. A condensate (28.9 g) was obtained which had a somewhat higher refractive index $n_D^{20}=1.4238$. Thus, the 32.4 g of the product of hydrogenation actually contained some cyclohexane which was converted into benzene. To eliminate the latter, the condensate obtained after dehydrogenation was treated with a 5% oleum, wetted, dried and distilled over sodium. The hydrocarbon passed over at 72—73°/772 mm leaving no residue. The index of refraction of this hydrocarbon was $n_D^{20}=1.4099$, specific weight $d_4^{20}=0.7476$; $M_D=27.86$; calculated for C_6H_{12}—27.71.

In a second experiment, after bringing cyclohexene twice into contact with aluminium oxide, the main fraction was obtained boiling at 70—74° which had a refractive index $n_D^{19}=1.4290$. This fraction (22.1 g) was hydrogenated as formerly and this condensate (17.7 g) had $n_D^{18}=1.4122$. After dehydrogenation 15.8 g were obtained which had $n_D^{19}=1.4169$. The latter after the elimination of benzene gave 12.6 g of a hydrocarbon, b. p. 71—72°/749 mm; $n_D^{20}=1.4094$; $d_4^{20}=0.7466$; $M_D=27.87$; calculated for C_6H_{12}—27.71. These physical constants leave no doubt about our having obtained methyl-cyclopentane from cyclohexene as a result of contact reactions. This hydrocarbon could have been formed only through the hydrogenation of m e t h y l-c y c l o p e n t e n e, which was the result of the contact transformation of cyclohexene. Thus, the latter has undergone an isomerization into a five-membered unsaturated cycle with a methyl side group.

In 1933 Zelinsky and Kasansky [2] have proved that cyclopentane hydrocarbons are converted through hydrogenation catalysis at a temperature of 300° into hydrocarbons of the paraffinic series. From monosubstituted cyclopentanes a mixture of paraffinic hydrocarbons was formed in which iso-forms prevailed.

After passing the methyl-cyclopentane obtained by us ($n_D^{20}=1.4094$) in a current of hydrogen over platinized silicagel at 250° at a rate of five drops per minute, the condensate obtained had $n_D^{20}=1.3751$. It was distilled over sodium and had the following properties: b.p. 61.5—63.5°/761 mm; $n_D^{20}=1.3749$; $d_4^{20}=0.6585$; $M_D=29.93$; calculated for C_6H_{14}—29.91. Thus, methyl-cyclopentane can be completely converted by means of this contact into a paraffinic hydrocarbon even at a temperature of 250°.

S i l i c a g e l a n d C y c l o h e x e n e. Cyclohexene (31.3 g) was passed once over silicagel at 450°, at the same rate as in the previous experiments. By distilling the condensate the following fractions were obtained:

1. 68—72°	0.2 g	$n_D^{20}=1.4263$
2. 72—76°	23.6 »	» $=1.4308$
3. 76—80°	2.3 »	» $=1.4344$
4. Residue	2.3 »	

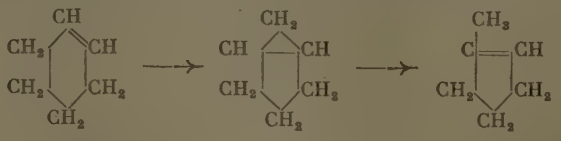
Fraction 72—76° (23.3 g) was hydrogenated in a current of hydrogen over platinized charcoal. A saturated condensate (21.5 g) was obtained that had a refractive index $n_D^{19}=1.4122$. This condensate was subjected to dehydrogenation at 300° and the product of dehydrogenation had a refractive index $n_D^{20}=1.4160$. After a treatment with oleum in order to eliminate benzene, a hydrocarbon (14.2 g) was obtained boiling at 71.5—72.5°/755 mm; $n_D^{20}=1.4089$; $d_4^{20}=0.7445$; $M_D=27.92$; calculated for C_6H_{12}—27.71.

After passing this hydrocarbon once over platinized silicagel in a current of hydrogen at 275° at a rate of 5 drops per minute, a complete cleavage of this ring was obtained. The condensate had $n_D^{20}=1.3749$ and after being distilled over sodium it had the following properties: b. p. 61.5—63.5°/760 mm; $n_D^{20}=1.3748$; $d_4^{20}=0.6581$; $M_D=29.94$; calculated for C_6H_{14}—29.91.

Thus, when cyclohexene was passed once over silicagel at a temperature of 450°, its almost complete transformation into methyl-cyclopentene was achieved. Experiments with silicagel were repeated many times, always with the same positive results. If cyclohexene was contacted at a lower temperature (400°), the fraction 72—76° of the condensate had a somewhat higher index of refraction $n_D^{17}=1.4338$, but the process went on in the same direction. Only a small amount of condensation products was formed and the main part of the latter distilled over in the interval from 200 to 250° and the distillate had $n_D^{20}=1.500$.

1-Methyl-cyclohexene-3 (b. p. 102—102.8°/741 mm; $n_D^{20}=1.4412$) being brought into contact with aluminium oxide and silicagel behaved in the same way as cyclohexene. In an experiment with silicagel at 450°, the condensate consisting of the transformation product gave a main fraction 92—105°, $n_D^{19}=1.4340$, which after hydrogenation over platinum was converted into a hydrogenated product having $n_D^{20}=1.4168$. Its dehydrogenation gave a product with $n_D^{20}=1.4321$ which was washed with oleum in order to eliminate toluene, and the remaining hydrocarbon had the following constants* corresponding to dimethyl-cyclopentane: b. p. 92—95°/755 mm; $n_D^{20}=1.4130$; $d_4^{20}=0.7508$; $M_D=32.58$; calculated for C_7H_{14}—32.33. Hence the immediate product of the transformation of methyl-cyclohexene was dimethyl-cyclopentene which after hydrogenation was converted into dimethyl-cyclopentane. After passing this dimethyl-cyclopentane over platinized silicagel in a current of hydrogen at 275° and at a rate of 5 drops per minute, a condensate was obtained having $n_D^{20}=1.3904$. After being distilled over sodium this hydrocarbon had the following properties: b. p. 86.0—93.0°/761 mm; $n_D^{20}=1.3900$; $d_4^{20}=0.6891$; $M_D=34.44$; calculated for C_7H_{16}—34.53. Thus, we have achieved a conversion form methyl-cyclohexene to dimethyl-cyclopentene and from the latter through dimethyl-cyclopentane to a mixture of isomeric heptanes.

The mechanism of the isomerization of cyclohexene under the influence of the contact may be expressed by the following scheme:



i. e. through the intermediate formation of bicyclo-(0.1.3)-hexane. The same-scheme should be adopted for the isomerization of methyl-cyclohexene into dimethyl-cyclopentene. From the point of view of chemical stability one

* See Beilstein, 5, 33, 34.

has to acknowledge that the cyclohexene molecule is less stable under conditions of its contact with the heated surface of aluminium oxide or silicagel than methyl-cyclopentene.

In order to become convinced that the ring system of cyclopentene is stable, we passed 23 g of cyclopentene over aluminium oxide at 450° at the same rate as in the experiments described above. Cyclopentene was obtained by the dehydration of cyclopentanol over the same aluminium oxide at 320° and had the following properties: b. p. 45.5—46°/777 mm; $n_D^{16}=1.4255$; $n_D^{20}=1.4229$. The condensate (22.8 g) was subjected to distillation and the distillate had the same properties as the initial hydrocarbon: b. p. 45—46°/776 mm; $n_D^{16}=1.4255$. The same results were obtained when cyclopentene was brought into contact with silicagel: b. p. 44.5—45°/750 mm; $n_D^{20}=1.4229$.

It is only cyclohexene that easily undergoes isomerization with the formation of a cycle having five carbon atoms.

In this respect it seemed interesting to find how cyclohexane would behave under the same conditions. Cyclohexane ($n_D^{20}=1.4269$) proved to be absolutely stable. When it was passed at a rate of 3—3.5 g per hour at 450° over aluminium oxide as well as over silicagel its properties did not change: b. p. 80.5—81°/750 mm; $n_D^{20}=1.4269$.

Hence it became absolutely clear that it is only the unsaturated character of cyclohexene that makes it an unstable hydrocarbon. The unsaturated character of cyclopentene does not hinder its remaining a stable compound at an elevated temperature when brought into contact with aluminium oxide and silicagel.

Chromium Oxide and Cyclohexene. Pure green chromium oxide precipitated by ammonium hydroxide from a solution of chromium nitrate has been investigated. The chromium oxide was previously dried and heated in a current of carbon dioxide up to 450°. Several experiments were staged at this temperature and all of them led to one and the same result. Cyclohexene ($n_D^{17}=1.4480$) was brought repeatedly into contact with chromium oxide at an average rate of 5 g per hour. After every passage, the refractive index of the condensate rose and an evolution of hydrogen was observed: $n_D^{21}=1.4592$; 1.4671; 1.4748; 1.4832.

After the fourth passage, the condensate was treated with an equal volume of 80% sulphuric acid in order to eliminate traces of cyclohexene that had undergone no change, washed with water, dried over calcium chloride and distilled over sodium. With the exception of a very small residue, the hydrocarbon passed over at 77—78°/745 mm and had a refractive index $n_D^{21}=1.4900$, i. e. it consisted mainly of benzene and contained traces of cyclohexane. The same process takes place over chromium oxide at a temperature of 400°, but only the process is a little less active.

Chromium Oxide and 1-Methyl-Cyclohexene-3. This hydrocarbon ($n_D^{20}=1.4412$) of which 12 g were taken, after being passed thrice over chromium oxide gave a condensate with a refractive index that had considerably increased: $n_D^{20}=1.4862$. After a treatment with 80% sulphuric acid the hydrocarbon passed over at 103—108° and had an index of refraction $n_D^{21}=1.4843$. 8 g were obtained. Thus, 1-methyl-cyclohexene-3 had been converted mainly into the corresponding aromatic hydrocarbon (toluene) which was almost totally absorbed after a treatment with a double volume of sulphuric acid monohydrate. A small hydrocarbon layer that remained ($\sim$1 g) was distilled over sodium and boiled at about 100°, having an index of refraction $n_D^{20}=1.4276$, corresponding to that of methyl-cyclohexane. Therefore chromium oxide brings about the dehydrogenation of methyl-cyclohexene which is accompanied by a parallel reaction of the hydrogenation of

a small part of its molecules at the expense of hydrogen eliminated in the dehydrogenation reaction.

Cyclohexane and Chromium Oxide. It has been found recently ([3]) that chromium oxide, preliminarily heated in a current of hydrogen at 450°, depending on the method of its preparation, may be a more or less active catalyst for the dehydrogenation of cyclohexane. Therefore it seemed important to find out the behaviour of cyclohexane when brought into contact with chromium oxide, that had not been preliminarily treated with hydrogen. It has been shown above that such a chromium oxide brings about the intensive dehydrogenation of cyclohexene and methyl-cyclohexene into benzene and toluene.

Cyclohexane ($n_D^{20}=1.4269$) of which 9.3 g were taken, was passed over such chromium oxide at a rate of 4.5 g per hour at a temperature of 450°. The condensate (8.9 g) was distilled over sodium and passed over at 80—80.5°/746 mm, having $n_D^{20}=1.4283$. This shows that chromium oxide, unless it is treated preliminarily with hydrogen, fails to bring about the dehydrogenation of cyclohexane.

Cyclopentene and Chromium Oxide. Cyclopentene (b. p. 45.5—46°/777 mm; $n_D^{20}=1.4229$) of which 9.7 g were taken, was brought under the same conditions into contact with chromium oxide. A condensate (7.5 g) was obtained that boiled over sodium at 43.5—44.5°/745 mm and had an index of refraction $n_D^{20}=1.4251$. Cyclopentene as well as cyclohexane proved to be very stable when they were brought into contact with chromium oxide at 450°.

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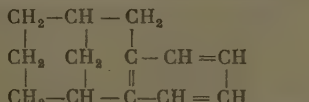
CHEMISTRY

ON THE CONTACT TRANSFORMATIONS OF BENZBICYCLONONENE

By N. V. ELAGINA and N. D. ZELINSKY, Member of the Academy

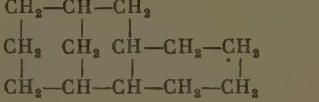
Lately Cook and Hewett (1) have shown that the cyclo-dehydration of the three isomeric carbinols—1-benzyl-cyclohexanol-1, 2-benzyl-cyclohexanol-1 and phenyl-cyclohexyl-carbinol—leads to the formation of one and the same saturated hydrocarbon benzbicyclononene which is a derivative of bicyclo-(1.3.3)-nonene. This hydrocarbon, the mechanism of the formation of which from the three isomeric carbinols remains obscure, presents considerable theoretical interest because of its extremely peculiar structure which has been proved by the above-mentioned authors. They have established that this hydrocarbon was not hexahydrofluorene. In the basis of its structure lays bicyclo-(1.3.3)-nonene, two carbon atoms of which have participated in the formation of the benzene nucleus in the tricyclic molecule of benzbicyclononene (I)



(I)

Cook and Hewett mark out that such a structure of benzbicyclononene must be the cause of a slight strain in the system of this hydrocarbon on account of the deviation from the normal direction of the valencies of two carbon atoms of bicyclononene taking part in the formation of the aromatic ring in this tricyclic molecule. Further, they point out that benzbicyclononene is not dehydrogenated by selenium, is passive towards platinum-black at 300°, and is not dehydrogenated by platinized asbestos even at a temperature of 340—360°. On the basis of our former investigations one was to expect theoretically that in contact with platinum or palladium benzbicyclononene should undergo hydrogenation or dehydrogenation catalysis, depending on the temperature. In the investigation of Zelinsky and Ivanov (unpublished as yet) it has been found that the bicyclo-(1.3.3)-nonene of Meerwein is dehydrogenated at 300° in contact with platinized charcoal, with the formation of propylbenzene. Therefore it was proved that notwithstanding the absence of strain in the system of bicyclononene and its relative stability, one of its six-membered rings undergoes dehydrogenation and the other breaks on adding two atoms of hydrogen. In this way an aromatic hydrocarbon is formed.

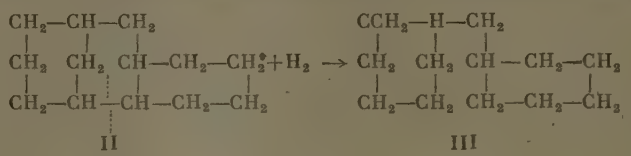
Benzbicyclononene was prepared by the cyclodehydration of 1-benzyl-cyclohexanol-1 with phosphoric oxide with a good yield. We proposed to obtain by the complete hydrogenation of benzbicyclononene hexahydrobenzbicyclononene (II):



(II)

To attain this, benzbicyclononene (b. p. 121—123°/14 mm; $n_D^{21}=1.5431$ and $d_4^{21}=0.9848$) was slowly passed over palladized charcoal, at first at 250° and then at 220°. After passing it several times, a condensate was obtained that had a much lower index of refraction $n_D^{20}=1.4957$. In order to eliminate benzbicyclononene that had not been reduced, the condensate was treated several times with monohydrate. The purified and completely saturated hydrocarbon had the following properties: b.p. 247—249°/743 mm; $n_D^{21}=1.4764$, $d_4^{21}=0.8769$. These constants showed that the hydrocarbon obtained by us had the composition $C_{13}H_{24}$ and not $C_{13}H_{22}$, as we had expected. The molecular refraction (found) was 58.00; calculated for $C_{13}H_{24}$, 57.83.

Thus, it is not six, but eight atoms of hydrogen that had added to benzbicyclononene and this had led to the rupture of one of the six-membered rings in the expected hexahydro-benzbicyclononene. A further investigation showed that the rupture had occured at a definite spot of the intermediate hexamethylene ring in the tricyclic system of hexahydro-benzbicyclononene which because of this rupture was converted into bicyclohexyl-methane (III).



II III

This fact presents some theoretical interest: in a condensed, completely saturated tricyclic system of 2.3-tetra-methylene-bicyclo-(1.3.3)-nonane (II), consisting of hexamethylene rings, one of them is easily broken under the influence of contact hydrogenation.

On the basis of our former data (2) bicyclohexyl-methane has the following properties: b.p. 250—252°/760 mm; $n_D^{20}=1.4752$; $d_4^{20}=0.8750$, $M_D=58.05$. The hydrocarbon obtained by the contact hydrogenation of benzbicyclononene had almost the same properties: b. p. 247—249°/743 mm; $n_D^{21}=1.4764$; $d_4^{21}=0.8769$; $M_D=58.00$.

Some time ago we have found (2) that bicyclohexyl-methane under conditions of dehydrogenation catalysis is converted into f l u o r e n e. In order to prove that the perhydrated product of the reduction of benzbicyclononene was actually bicyclohexyl-methane we brought it into contact with platinized charcoal at 300° and became convinced that by losing 14 atoms of hydrogen it was w h o l l y converted into fluorene, m. p. 114°.

Thus, we have proved that the original tricyclic molecule of benzbicyclononene could be easily converted through contact hydrogenation by passing an intermediate phase—hexahydrobenzbicyclononene—into bicyclohexyl-methane which in the process of dehydrogenation contact gives the tricyclic system of f l u o r e n e.

Without having recourse to vigorously acting reagents, we have realized all these transformations in the presence as well as in the absence of free hydrogen by a simple contact with platinum or palladium.

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CHEMISTRY

SYNTHESIS OF TIN HYDRIDE BY THE ACTION OF ATOMIC HYDROGEN UPON TIN CHLORIDE

By V. M. VDOVENKO

(Communicated by V. E. Tishtchenko, Member of the Academy, 5. IV. 1939)

The present work is an attempt of obtaining tin hydride by the action of atomic hydrogen upon tin chloride. This reaction has been investigated qualitatively by Kroepelin and Vogel [1] who observed a tin mirror to form on the glass tube. The authors explain the formation of the metallic mirror by the decomposition of SnH_4; yet the presence of tin hydride in the discharge tube was not detected by them. We deemed it wise to ascertain whether tin hydride is actually forming on this reaction and if so to determine its yield quantitavely.

Active hydrogen was obtained in a U-shaped glass tube, 35 mm in diameter and 1.5 m long, provided with aluminium electrodes. The source of high tension was a 2 kW power transformer. The electrodes of this tube carried an alternating voltage of 6000 Volts. Molecular hydrogen, obtained by the electrolysis of sodium hydroxide solution between nickel electrodes, was passed through a tube filled with platinized asbestos, and while placed inside an electric furnace heated to 350—400°. The hydrogen went then through sulphuric acid into reserve baloons provided with gauges, from where through a gas-bubble counter filled with phosphoric acid it passed into the discharge tube. The inside of this tube before the experiments was strongly wetted with phosphoric acid which, as is known, lessens in a great degree the recombination of hydrogen atoms. From the tube the mixture of active and molecular hydrogen was passed into a reaction vessel 50 mm in diameter which was cooled by liquid air. Tin chloride was pressed into tablets (10 mm in diameter, 4—7 mm broad) and introduced on a platinum net placed in a glass tube, into the reaction vessel. The pressure in the tube was 0.3—0.4 mm Hg. The amount of hydrogen passing through the tube was 0.25—0.30 liter per hour (measured at 760 mm and 20°). Care was given to secure equal conditions throughout the experiment.

From the very start we found active hydrogen to reduce tin chloride to pure metal. The crystals of tin chloride became covered by a dark brown layer in which tin regulus could clearly be seen. At the same time the walls of the glass tube were coated with a thin metallic tin film whose density decreased along the length of the tube.

We here refer to the data obtained in one of the many tests we have carried out. In 60 min. the weight of tin chloride sample (0.69 g) was diminished

by 0.20 g. The weight of the metallic coating was 0.03 g. In the course of the experiment the crystals of tin chloride were being coated with a thin layer of metallic tin and the reaction between the atomic hydrogen and the salt was slowed down. A great quantitative difference was also observed between the weight lost by the sample and that of the metallic coating. It would seem that an essential part of tin chloride, reacting with atomic hydrogen, was converted into hydrogen chloride and metallic tin left in the vessel.

Besides, tin chloride may be supposed to have volatilized under the action of heat produced by the recombination of hydrogen atoms.

It was thought important to establish experimentally whether the entire tin hydride is decomposed forming a tin mirror or it could be partly obtained by being condensed in a reaction vessel cooled with liquid air. When the discharge had expired, the apparatus was filled with hydrogen and with the help of a glass tube was connected by a rubber tube with two broad testing-tubes filled with an aqueous solution of Ag_2SO_4. After hydrogen was passed through the whole system for a run time of 20—30 min., the Dewar vessel with liquid air was very slowly taken away. Simultaneously tin chloride was progressively evaporating and passed with hydrogen into the test-tubes filled with silver sulphate. As the mixture of hydrogen with tin hydride entered the tubes, the colourless solution of silver salt grew darker and ended with precipitating a characteristic black-brown residue [according to Haken ([2]), of the composition, approximately, $SnAg_4x\ AgNO_3$] in which, after filtration, the tin content was estimated.

These are results obtained by acting with active hydrogen upon tin chloride (Table 1).

Table 1

Weight of tin chloride tablet in g	Test time in min.	Weight of metallic coating in g	Quantity of tin found in residue in micrograms
1.714	60	0.026	traces (5)
1.490	60	—	75
1.183	60	0.0089	34
1.576	60	—	89
1.115	80	0.021	75
1.411	60	0.015	55
		Mean	55

The data given in Table 1 show that the formation of tin hydride really takes place. The yield, under the conditions provided, is ranging within 30—90 micrograms and in the average is 55 micrograms. It should be noted that the conditions of obtaining tin hydride were unfavourable, as the mirror of metal on the glass walls of the apparatus is apt to accelerate the decomposition of the hydride.

Paneth ([3]) on adding certain organic compounds to hydrogen observed a rise in the yield of SnH_4 under conditions of electric discharge. We attempted therefore quantitatively to investigate the influence of methane upon the yield of SnH_4.

Table 2 shows the results obtained by acting upon tin chloride with a mixture of active hydrogen and methane.

As seen from Table 2, small additions of methane essentially raise the yield of the hydride which in some cases amounts to 250—280 micrograms per hour. The average increase in yield is from 55 to 150 micrograms per

Table 2

Weight of tin chloride tablet in g	Test time in min.	Gas	Weight of metallic coating in g	Quantity of tin residue in micrograms
1.916	60	H_2 + 0.1%CH_4	0.013	240
2.014	60	H_2 + 1%CH_4	0.010	144
2.136	60	H_2 + 1%CH_4	0.008	226
2.008	60	H_2 + 2%CH_4	0.013	75
2.010	60	H_2 + 2%CH_4	0.015	116
2.136	60	H_2 + 3%CH_4	0.011	281
2.101	60	H_2 + 4%CH_4	0.013	137
2.104	60	H_2 + 5%CH_4	0.013	89
2.156	60	H_2 + 7%CH_4	0.010	130
1.613	60	H_2 + 10%CH_4	0.011	68
			Mean	150

hour. Addition to molecular hydrogen of large quantities of methane (up to 10%) failed to increase the yield of tin hydride.

Acknowledgement is greatfully made to Prof. S. A. Schukarev for advice and continual interest he has taken in the work.

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CHEMISTRY

BEHAVIOUR OF PHOSPHORIC ACID SOLUTIONS AS IDEALLY CONCENTRATED SOLUTIONS

By I. D. YUDIN

(Communicated by Wl. A. Kistiakowsky, Member of the Academy, 15. III. 1939)

Papers by Nernst (1), Brönsted (2), Rümelin (3), Katz (4), Vrevsky and Nikolsky (5) have been dedicated to the study of ideally concentrated solutions.

Different authors give different kinds of equations to express the properties of ideally concentrated solutions, thus, for instance, Nernst gives the equation:

$$\left(\frac{\partial Q}{\partial N}\right)_{N=N_2} - \left(\frac{\partial Q}{\partial N}\right)_{N=N_1} = RT \ln \frac{P_1}{P_2} = A, \tag{1}$$

where Q is the solution heat effect when one grammole of water is added to an infinitely large quantity of solution with a concentration of N moles of water to one mole of solute, P_1 and P_2 are the vapour tensions of the 1st and 2nd solutions respectively.

Brönsted proposes an equation of the following type:

$$\frac{dU}{dN} = \frac{dA}{dN}, \tag{2}$$

where U is what Brönsted calls differential heat of separation. By means of this equation Brönsted emphasizes the fact that U and A are equal at only one point.

Vrevsky and Nikolsky propose the following inequality:

$$\left(\frac{\partial Q}{\partial N}\right)_{N=N_2} - \left(\frac{\partial Q}{\partial N}\right)_{N=N_1} > A_2' - A_1', \tag{3}$$

where $A_2' = RT \ln \frac{P_0}{P_2}$, $A_1' = RT \ln \frac{P_0}{P_1}$, P_0 is the vapour tension of pure water.

In the papers of the authors mentioned above H_2SO_4 solutions were studied, and although Rümelin and Katz studied phosphoric acid solutions as well, their data are contradictory.

This fact gave us the idea to begin a supplementary study of H_3PO_4 solutions. Besides, such a study is interesting also because, to our opinion, it shows one of the possible methods of practically checking the III law of thermodynamics (Barthelot-Nernst's law) which was formulated more generally by Wl. A. Kistiakowsky (6).

The value we chose by which completely to characterize the full energy change in the system, was the differential heat of dilution, i. e. the heat effect produced when one grammole of water is added to an infinitely large quantity of solution. In our work we kept to the principle of Nernst and Orthman (7), Roth and Frau (8) and Rümelin (9).

The apparatus was a differential compensational calorimeter which consisted of two identical calorimetric vessels*. A calorimetric chamber with calorimetric tubes inside of its, thermoelement, heaters and a stirring rod were put into a double walled thermostat.

The mutual arrangement of all the parts of the apparatus can be seen from Fig. 1.

The temperature of the thermostat was kept constant with accuracy to 0.01° C by means of a mercury-toluol regulator, fashioned after Magnus' diagram.

Before the experiment the phosphoric acid was kept in the thermostat for 2 hours at 25°C and then was distilled by means of a siphon into the calorimeters. The aqua value of the system was determined after the electric method (10). The heater was a constantan wire $d=0.1$ mm, melted into a glass capillar and bent in spiral form, according to Lange. The length of the capillar 40 cm, the resistance of the heater—28 Ohms. The current source was an accumulator battery of 12 V. A normal 10-Ohm Siemens-Galske resistance coil. The zero-instrument was a Hartman-Braun mirror galvanometer. The current flow time was determined by means of a contact chronometer. The compensator was fed with a current of 0.01 A. Two multipliers whose parts gave the necessary voltage to the compensator were switched in parallel to the resistance of the heater. Taking into account the arms of the multipliers and beginning with the equation

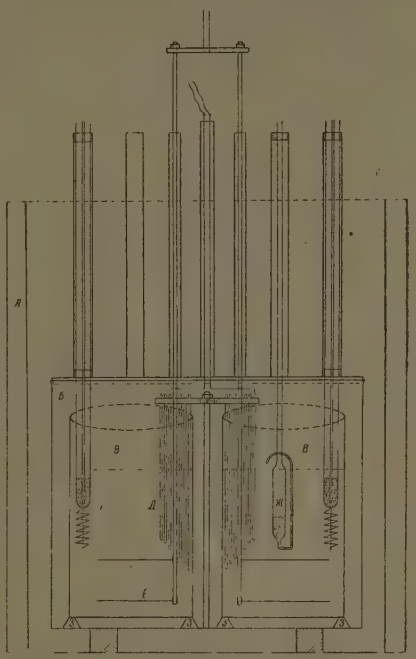

Fig. 1. A—thermostat, $Б$—calorimetric chamber, B—calorimetric vessels, $Г$—heaters, $Д$—thermoelement, E—stirring rod, $Ж$—Steinwer pipette, $З$—ebonite stands.

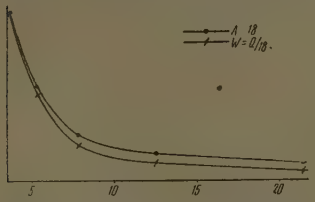

Fig. 2. Heat of dilution and maximum operation of H_3PO_4 solutions ot 25°C (according to the author's data)

$$Q = 0.2389 \frac{E^2}{R_1'} t \qquad (4)$$

we deduct the following equation:

$$Q = (e_1 e_2 A - e_1^2 B) t \qquad (5)$$

where e_1 and e_2 are the tensions on the ends of the heater and the etalon resistance, t is the time in seconds, A and B—constants depending on the arms of the multipliers.

A sensitivity test of the thermoelement system showed that 1 mm on the

* The diagram of the calorimeter is taken from Rümelin.

scale of the mirror galvanometer, under our conditions, corresponded to a temperature difference of 0.00008° C. We introduced the water into the calorimeter after Steinwer's method.

The heat effect of dilution was determined after the usual thermochemical method. The correction for radiation was calculated by Regnault-Pfaundler's formula([12]). The correctness of the apparatus' readings were checked by sulphuric acid. The work was calculated according to the formula:

$$A = RT \ln \frac{P_0}{P_1}. \qquad (6)$$

The vapour tension of the solution (P_1) was calculated by the formula of a parabola of the fourth order ([12]):

$$P_1 = P_0 + ax + bx^2 + cx^3 + dx^4, \qquad (7)$$

where x denotes the concentration of the solution in per cent and a, b, c are constants. The average percentage error in determining the differential dilution heats of H_3PO_4 solutions is 1.06%; this value is too insignificant to have had any effect on the curve $\frac{dQ}{dN} = f'(N)$ (Fig. 2).

In order to compare the differential dilution heats and maximum works calculated for a mole of water, we are including table and a curve (Fig. 2).

Concentration in %	N	P	$Q/18$	$A/18$	$Q/18 - A/18$
60	3.561	12.671	20.385	20.423	0.038
50	5.273	16.730	10.474	11.503	1.029
40	7.760	20.486	4.375	5.462	1.087
30	12.530	21.464	2.217	3.297	1.080
20	21.490	22.212	1.078	2.201	1.123

The curve (Fig. 2) expresses the relation between the dilution heat and the maximum work of phosphoric acid solutions and N at 25°C, where N is the number of moles of water per mole of acid, P is the vapour tension of water over a H_3PO_4 solution of given concentration. As Fig. 2 shows, the curves $\frac{A}{18} = f(N)$ and $\frac{Q}{18} = f'(N)$ which were constructed on the basis of data obtained by us, are very close to one another all along the interval examined by us for dilution.

However, neither strict parallelizm nor merging can be observed in the curves. For weaker solutions beginning with $N = 21.44$ to $N = 7.76$ the curve $\frac{A}{18} = f(N)$ is above the curve $\frac{Q}{18} = f'(N)$ and gradually comes closer to it, passing over to the region where $N = 7.76$ decreases to $N = 3.56$, the curve $\frac{A}{18} = f(N)$ approaches the curve $\frac{Q}{18} = (f'N)$ more quickly, and finally at the point where $N = 3.56$ both curves practically merge.

An analogous relation was given for H_2SO_4 in Vrevsky and Nikolsky's paper, based on both their own data and on that of Rümelin and Brönsted.

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GEOLOGY

REGULARITIES OF DEVELOPMENT OF CLOSED BASINS

By B. P. KOLESNIKOV

(Communicated by A. D. Archangelsky, Member of the Academy, 14. IV. 1939)

There are basins freely communicating with the ocean, basins with obstructed communicaton and lastly closed basins. Similar basins existed also in the past.

Particular interest belongs to closed basins. Their economic importance is very high. For instance, the Caspian Sea with its fisheries and its natural mirabilite factory—the Kara-bogaz-gol. In the sediments of Tertiary closed basins of the Pontian-Caspian region many useful minerals are encountered (oil, iron ores, various salts etc.).

Naturally enough, much attention has always been given to the study of the stratigraphy of such sediments. The stratigraphic scheme of the Upper Tertiary sediments is one of the most detailed. And yet search and prospecting work requires a more and more detailed stratigraphy to be worked out. Oil companies conduct extensive work in this direction. Detailed schemes are being worked out for individual, usually very small regions. As a rule, it is impossible to correlate such schemes. This is accounted for by the fact that the work aims only at establishing marking horizons and leading forms. As the conditions of sedimentation and habitation are different in different often closely spaced portions of the basin, the ensuing stratigraphic schemes are different and difficult to compare, too.

The only means to regulate the stratigraphy is to put the stratigraphic work on a serious theoretical basis. It is necessary to strive to show the regularities of the development of the basins and their fauna. Only then the stratigraphic scheme will cease to be conventional when it will reflect the change of events. The stratigraphic scheme is to be deduced from geological history.

For such work the sediments of closed basins offer an especially suitable material. Such basins are kind of small natural laboratories where those processes which would require considerable time in the open sea, develop at a high rate.

Closed basins were susceptible to all changes of the physico-geographical conditions, especially to the movements of the earth's crust. While in basins freely communicating with the ocean the movements of the earth's crust were manifested chiefly in the configuration of the shores and in shift of facies, in isolated basins they exerted a much deeper influence. In fact it may be said that the movements of the earth's crust essentially determined the conditions of closed basins.

The history of the Neogene seas of the Pontian-Caspian region abounds in examples testifying to the intensification of movements in mountainous regions accompanied by a development of piedmont depressions where accumulated thick beds mostly of shallow water sediments. The development of such depressions greatly affected the closed basins.

During the epochs of intensification of movements of the earth's crust considerable masses of water accumulated in the piedmont depressions; the sea, or rather the lake-sea began to regress (see the scheme, the Upper Sarmatian, Meothis, the Balakhanian stage, the Lower Apsheronian, the Baku stage); the area of evaporation diminished and the salinity of the upper horizons of water became lower. In general the conditions of the whole basin changed and this in turn affected the fauna which was very susceptible to every change. Being unable to withstand the sharp changes of conditions during such epochs, the fauna died out partly (Upper Akchagyl and Upper Apsheronian) or completely (Upper Sarmatian and Balakhanian).

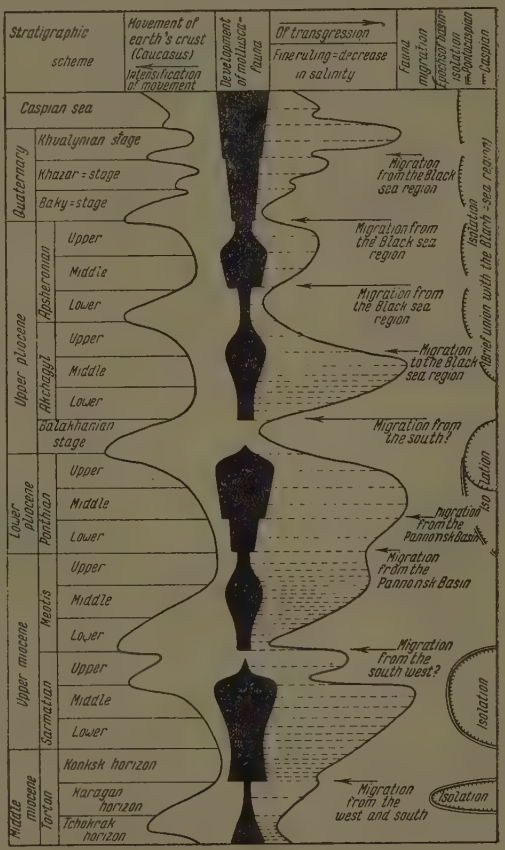

Fig. 1.

At relatively calm epochs the fauna adapting itself to the changing conditions started variating and nearly each species gave a group of forms differing from one another to such an extent that when approaching their taxonomy purely iconographically, almost each shell would have to be described either as a new species or as new variety. Unfortunately, it is often done (the Baku stage, the Caspian Sea, etc.).

During calmer epochs, i. e. at epochs of slight movements, a transgression of closed basins usually took place with a luxurious development of mollusca (Middle Sarmatian, Middle Apsheronian). The variability of species sharply decreased. If such conditions lasted for a long period of time, the development of fauna proceeded in a somewhat different direction. Many species began to die out being supplanted by those better adapted for life under these conditions, and the number of individuals of these species increased (Upper Apsheronian).

The alteration of closed basins should no doubt finally lead to a complete dying out of the descendants of marine forms, if the population of lake-seas were not filled up by newcomers migrating hither during the epoch of disturbed isolation (see scheme).

The observed regularities of development of Tertiary closed basins naturally suggest the idea of determining the phase of development of the present time Caspian Sea. During the post-Baku time the outlines of the Caspian basin were subjected to frequent alterations. Some of these were possibly due to glaciations, but the pre-Khvalynian regression is no doubt connected with intense movements at the Caucasus. The Khvalynian which was a comparatively calm epoch was marked by an extensive transgression. During the post-Khvalynian time the Caspian basin regressed to a considerable extent, this being evidently connected again with an intensification of the movements. This process is continuing up to the present time. Thus, when calculating the changes of the Caspian Sea level it is necessary to consider also the tectonic factor. At present the material available is not yet sufficient, but we expect that a further study of the fauna—the most susceptible sign of the alterations of the basin—will enable us to throw light on this important question.

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GEOLOGY

TECTONICS OF THE NORTH ARAL REGION

By A. L. YANSHIN

(Communicated by A. D. Archangelsky, Member of the Academy, 16. IV. 1939)

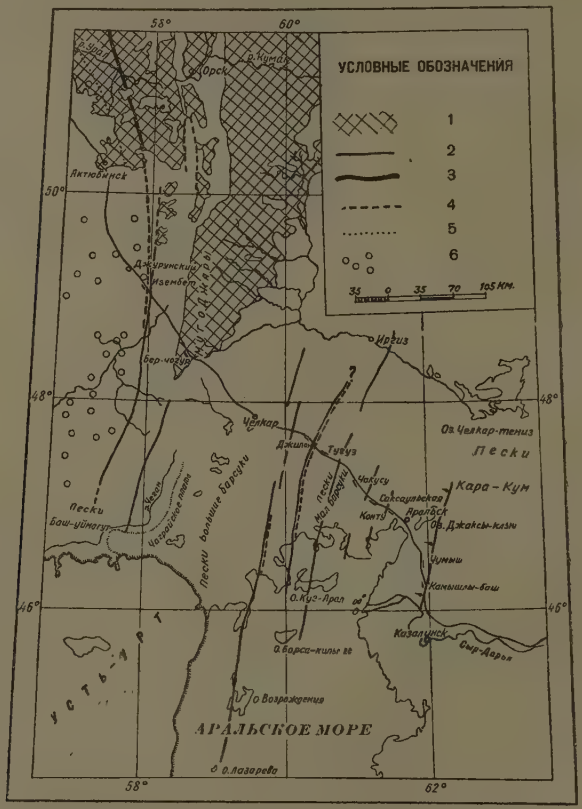
As a result of a three year work carried on by the Academy of Sciences, a system of gentle folds of Uralian direction was found to exist in the North Aral region, this system involving also the Chushkakul anticline, the dislocations of the Kulanda peninsula and a number of other dislocations hitherto unknown.

In the west the folded zone is limited by a flexure line crossing the upper reach of the right-hand tributaries of the Ilek river and the station Izembet, to unite further south with the steep western limb of the first anticline. This line is a structural continuation in the Mesozoic of the belt of the ancient Paleozoic thrust over the Anthracolithic foot-hills of the western slope of the South Urals. Further north in the section along the Sakmara river the width of the belt of Carboniferous and Lower Permian deposits from the line of these overthrusts in the east to the boundary with salt-bearing deposits of the Kungurian in the west is 45 km; at the latitude of the town of Aktyubinsk it decreases to 20 km; at the station Izembet, however, and further south the dome-like and brachy-anticlinal folds of Mesozoic rocks testifying to the presence of salt-masses at the depth, approach directly the line of flexures. This allows us to assume that southward from the upper reach of the Ilek river the overthrusts of the western slope of the Urals do not die out under the sheet of Mesozoic deposits, but, on the contrary, their amplitude becomes wider and brings the ancient Paleozoic directly in touch with the salt-bearing Kungurian deposits, as it is the case in front of the northern end of the Ufa plateau.

The first (from the west) anticline of the folded zone seems to be a continuation in the Mesozoic of the anticlinal belt of the crystalline schists of the South Urals. It crosses the Emba river up-stream from the mouth of its tributary, the Kum-dzharghan river, and is clearly seen at the latitude of the southern termination of the Mugodzhar in the basin of the Ata-dzhaksa river. The core of the fold is composed of sands of the continental Albian which is covered at the limbs by an incomplete series of sediments of the marine Upper Cretaceous and at some places directly by the Paleocene. The continuation of this anticline is to be sought at the foot of the Ustyurt in the sands of Teresken and Bash-uimaghut, which evidently represent in a certain portion the scattered chalky rocks of its core.

The Chushkakul anticline is lying further east beyond the meridional downwarping stretching along the western foot of the Mugodzhar and pro-

ceeding southward to the lower part of the Cheghan river. It is a direct continuation of the main western range Mugodzhar, issues from the place of its plunge with a strike of 185—200° and dips under the Tertiary deposits of the Ustyurt plateau between the Kenkous locality and Cape Chagrai.



Scheme of the tectonics of the North Aral Region.
Symbols: *1*—Paleozoic massifs. *2*—Axes ot anticlines. *3*—Shift of the western slope ot the Urals. *4*—Tertiary faults and flexures. *5*—Boundary of Artin and Kungur deposits. *6*—Salt domes.

The core of the anticline is throughout (for about 120 km) composed of continental beds of the Lower Cretaceous which are overlain by beds of marine sediments of the Upper Cretaceous and the Paleocene, transgressing from south to north and showing less intense dislocation. In the section the anticline has an asymmetrical structure with a steeper western limb complicated by flexure-like bents of the layers and small faults. The angles of the layer-pitch at the limbs usually equalize 3—5°, though at the western limb they occasionally attain 35°.

Beyond the Chushkakul anticline is lying a flat-dipping depression of a corresponding size whose axis passes from the station Chelkar to the northwestern corner of the Aral Sea. In its core beds of continental Oligocene and of Lower Miocene are lying at very low marks; while judging by the boreholes, cretaceous deposits subside below the sea level.

Behind the Bolshiye Barsuki sands, the band of which structurally corresponds to the outcrops of continental Oligocene at the eastern limb of the Chelkar anticline, there is stretching another anticline, a sloping uplift whose core is composed of sands of the Middle Eocene, and the limbs of an incomplete series of higher horizons of the Paleogene, each of these being less thick than usual. Against the background of this uplift a number of small brachy-anticlinal folds are to be distinguished whose cores show outcrops of clays of the lower series of the Aral Paleogene (their age has not been definitely established). The beds at the limbs of the brachy-anticlines show inclines up to 25°. In the southern part the roof of the main uplift is dissected for 60 km by a longitudinal fault with the eastern limb sinking to 80 m.

In the south the anticline turns slightly westward and continues (having passed under the Tsche-bas bay) in the dislocations of the Kulandy peninsula. Here a brachy-anticline with a strike of the axis to the NE at angle of 28° has been detected. Its core is composed of thick beds of the continental Mesozoic, and the limbs parting southward—of the Upper Cretaceous and the Nummulatic limestones of the Middle Eocene. The fold is complicated by small faults, orientated chiefly to the north-west, 320°, i. e. obliquely with respect to the axis.

Judging from recent hydrographic works, a stony shelf is stretching southward from Cape Uzan-cair at the Kulandy peninsula, along the bottom of the Aral Sea over a large distance, tracing the continuation of the ridge of the Maastrichtian limestones of the western limb of the Kulandy anticline. On the other hand, new hypsometric maps of the Vozrozhdenye Island make it possible to state that the principal strikes of the beds are meridional, more precisely orientated at 5—10°. These data testify to a direct connexion between the anticlinal uplifts of the Kulandy peninsula and the Vozrozhdenye Island, as assumed by A. D. Archangelsky ([1] and oth.). Further south in the same belt is located the Lazarev Island where limestones of the Maastrichtian stage dipping westward are found.

The anticline of the eastern margin of the Bolshiye Barsuki sands is followed by a narrow and shallow syncline filled with the continental Oligocene, and beyond this comes a new belt of uplifts whose anticlinal character is obscured by a large fault; this fault crosses the rail-road between the Dzhilan and Tuguz stations with a strike 20°, and farther south it acquires a meridional direction and stretches southward almost to the very shore of the Paskevich bay. The fault was traced over a distance of 140 km, its termination in the north being unknown. The subsiding limb of the fault is the eastern one; judging from facial character and the thickness of the series at both limbs, it seems to have longitudinally cleft the roof of the pre-existing anticline. The beds of the uplifted western limb of the fault southward from the rail-road line form a semi-dome, in whose core outcrops of the marine and continental Cretaceous are observed. The amplitude of the fault opposite the dome is up to 120 m.

The next anticline is marked by the meridional belt of the Malye Barsuki sands which present chiefly scattered deposits of the Middle Eocene sinking to a considerable depth westward and eastward. Further north, along the old high-road to Middle Asia, outcrops of Middle Eocene quartzites with *Pectunculus aralensis* Rom. are locally encountered. Still further north, at the continuation of these uplifts near the lake Kara-kul a bore-hole is located in the neighbourhood of the town of Irghiz, cutting diorites at a depth of 60 m.

Beyond the Malye Barsuki sands the type of folding changes. Small brachy-anticlinal folds are prevalent here which cannot always be united into

uplift zones. Such folds are observed in the western and north-eastern parts of the Kokturnak peninsula between the Chokusu and Saksaul stations and eastward from the latter (Kontu sands). In their cores outcrops of sands of the Middle Eocene and clays of the lower series of the Aral Paleogene are observed.

The axis of the marginal eastern syncline of the North Aral region is passing through the town of Aralsk. The Middle Eocene deposits have been encountered here in the course of boring at a depth of 107.70 m from the surface. Further east in the region of the lake Dzhaksy-klych and the stations Chumyshkul and Kamyshly-bash an uplift of the layers takes place, leading to the appearance at the surface of the Middle Eocene and of more ancient series of the Paleogene. It remains still obscure whether this uplift is connected with a new series of some local uplifts of the Aral region type or with the approach of Cretaceous rocks to the northern termination of the uplift, which according to the observations of B. A. Petrushevsky stretches from the Paleozoic Kara-tau range to the region of the Aral Kara-Kumy.

There is no doubt that the folded system of the Aral region is connected with the meridional system of folds of Cretaceous deposits in the basin of the Tobol river, established by P. L. Bezrukov ([2]), through the unexplored region of the lower part of the Irghiz river and the area between the latter and the Ulnoyak river. Less clear is the relation of the Aral region dislocations to the zone of sloping and extensive basins divided by transverse uplifts, found by G. E. Bykov ([3]) and B. A. Petrushevsky ([6]) in the eastern portion of the Turgai lowland. It is possible that against the background of these extensive and comparatively young downwarpings embracing different tectonic elements of the Hercynian base, a minute folding of the Aral type is preserved. Here such a folding must be very slanting and slightly manifested, inasmuch as stratigraphically the erosional cut is not deep here and only the upper horizons of Tertiary deposits are developed at the surface slightly dislocated also in the Aral region.

The general Uralian direction of the folds of the North Aral regions, the exact concurrence of the strike of the Chushkakul anticline with the strike of the anticlinal zone of the western Mugodzhar range whose direct continuation to the south it represents, the shallow depth of occurrence of the Paleozoic in the zone of uplifts near the town of Irghiz and the sharp contraction of the section of Mesozoic deposits in the anticlines, established by deep boring along the rail-road line—all these suggest the idea that the dislocations of the North Aral region have inherited their plan from the Hercynian epoch of orogenesis, that they develop the structural forms of this orogenesis and that at comparatively shallow depth Paleozoic rocks are occurring here throughout the cores of the anticlines.

When comparing the sections of the anticlines of the North Aral region with the sections of deep bore-holes sunk in the synclines, it is easy to see that the formation of the present elements of the tectonics of this region was intense as early as during the Cretaceous. The thickness of all the horizons of the continental and the marine Cretaceous at the anticlines is several times less than that at the synclines. Traces of intense erosion are observed at the anticlines among variegated clays, arbitrarily referred to the Aptian and the Albian, between the Albian and the Cenomanian, the Maastrichtian and the Paleocene.

In the Tertiary deposits both traces of erosion manifested in pebblestones and rough contact surfaces and disconformities connected with the folding are observed: 1) previous to the Middle Eocene; 2) between the Middle and Upper Eocene; 3) between the Lower and Middle Oligocene (intense); 4) near the Mugodzhar—previous to the deposition of the uppermost series of con-

tinental Oligocene, by its age corresponding probably to the Aquitanian (very sharp); 5) in the Middle Miocene (the formation of faults took place at the same time); 6) during the post-Sarmatian time. Even the sediments of the Aral-Caspian transgression are lifted at the anticlines up to 8—10 m above the sea level and sink below the sea level at the synclines. Northward from the sea shores there may be observed a similar behaviour of the Quaternary terraces of „sai" and drying rivers.

Thus, the present dislocations of the North Aral region whose plan corresponds to the tectonic elements of the Hercynian folded base, were developing throughout the known Mesozoic and Cenozoic history of this region.

The above data make us reconsider the existing conception as to the North Aral region and the adjoining parts of the Turgai lowland, as an almost immobile stable Hercynian platform which during the post-Paleozoic time was subjected aside from a general downwarping only to small chiefly fault dislocations, not conformable with the direction of the ancient folds ([4] and oth.). It is evident that from the tectonics of the covering deposits this region is essentially different from such regions of the earth's crust as the East-European pre-Cambrian platform. Elements of the Paleozoic structural plan corresponding to the pre-Cambrian structures of the East-European platform continue to exist and to develop here, determining the direction of all young movements. This bears evidence in favour of a considerable mobility of the folded base.

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GEOLOGY

NEW DATA ON THE UPPER TRIASSIC OF NORTH CAUCASUS AND THE CRIMEA

By A. S. MOISSEEV

(Communicated by A. A. Borissiak, Member of the Academy, 20. IV. 1939)

In the basins of the Sakhrai and Khodzh rivers at the North Caucasus the Lower Lias sandstones with bands of fine conglomerates and shell-rock discordantly overlie dark-red limestones with *Spirigera oxycolpos* Emm., *Sp. manzavini* Bitt. and others, belonging to the Rhaetic and the Norian stages. Lower down these rocks pass to light-coloured, pinkish and red massive limestones with *Pseudomonotis caucasica* Witt., and then to dark red and dark grey limestones with corals and crinoids belonging also to the Norian stage. The whole of this complex is underlain by Carnian clayey shales, banded sandstones and conglomerates with *Halobia austriaca* Mojs., *Halobia* aff. *superba* Mojs., *Arcestes*, etc.

Aside from brachiopods the Rhaetic and Norian limestones are characterized by the presence of abundant microfauna (*Foraminifera*, needless of sponges, *Ostracoda*, etc.), sponges (*Molengraffia regularis* Vinassa de Regny, *Sahraja triassica* gen. et sp. nov., *Steinmannia* sp., *Hodsia caucasica* gen. et sp. nov.), corals (*Thecosmilla labaensis* nov. sp., *Th. charliana* French., *Th.* ex gr. *subdichotoma* Volz., *Astraemorpha* ex gr. *confusa* Reuss, *Thamnastraea robinsoni* nov. sp., etc.), hydroid polyps (*Cherkesia robinsoni* gen. et sp. nov., *Circopora caucasica* nov. sp., *Heterastridium* sp.), bryozoans. An important rock-forming rôle belongs to the bushy algae *Corallinaceae* (*Lithothamnidium marii* gen. et sp. nov.), *Spongiostromata*, etc.

After a comparative study of the Upper Triassic of the Caucasus and the Crimea, the stratigraphic scheme of the Upper Triassic of the latter may be represented as follows for the Salghir river valley near the town of Simferopol and the Alma river valley up the village of Sablov. In the Salghir river valley the sandstones of the Lower Jurassic (Eskiordian series) pass below to sandstones and conglomerates of the Rhaetic among which bands of shale with *Halobia* and lenses of grey limestones have been encountered, a part of which has been deposited simultaneously with the enclosing rocks. The limestones enclose *Spirigera oxycolpos* Emm. and other forms characteristic of the Kessenbeds (Rhaetic). Besides, occasional *Pseudomonotis caucasica* Witt. have been encountered.

The next horizon of the Upper Triassic is composed of limestones, arranged in blocks in the Alma river valley. One of the blocks of grey and reddish

limestone up to 50 m in thickness which may be traced along the strike for 100—150 m lies among the taurician non-fossiliferous shales. At one place the eroded surface of the limestone and an overlap evidently of a Lower Jurassic conglomerate have been observed. This block has been found at the left-hand slope of the Alma river valley to the NW from the village of Beshui [in front of the forester's house (3)].

In the limestones are accumulated algae resembling those from Caucasus and the following fauna: *Spirigera oxycolpos* Emm., *Sp. manzavini* Bitt., *Rhaetina gregaria* Suess., *Pseudomonotis caucasica* Witt., etc.

According to its lithological composition and fauna this limestone is similar to the Rhaetic and Norian limestones of the Caucasus.

Near the above-described block, a block of grey limestone has been found, washed out of shales and filled with *Pseudomonotis caucasica* Witt., like that at the Caucasus. In the Crimea both these blocks should be regarded evidently as remnants of formerly thick limestone beds, eroded towards the end of the Rhaetic and at the beginning of the Lower Jurassic. By their age and stratigraphic position the limestone blocks of the Alma river are closely approaching though not identical with the lenses of limestone from the lower portion of the Eskiordian stage. Judging by the fauna they should, probably, occur below the latter.

The next horizon of the Upper Triassic is composed of shales with *Halobia* aff. *neumayri* Bitt. (similar to the Caucasian *H.* aff. *superba* of N. V. Robinson), *Pseudomonotis* and *Arcestes* (village of Eski-Orda and near the Totakoi eruptive in the Salghir river valley).

According to its stratigraphic position, this horizon corresponds at the Caucasus to the Carnian shales. The Crimean *Pseudomonotis*, like those from Caucasus, have been originally described as *Pseudomonotis ochotica* Tell., later on they have been isolated for the Caucasus as a new species *Ps. caucasica* Witt. which has been adopted by the present author also for the Crimean *Pseudomonotis*, originating from clayey shales.

The Crimean *Pseudomonotis caucasica* originating from clayey shales, is distinguished from typical Caucasian specimens from limestones by thinner costae of the first and second order. Allowance being made for the different stratigraphic position and their origin from different facies, the Crimean form is to be isolated as an independent variety under the name of *Pseudomonotis caucasica* nov. var. *taurica* [see (1), holotype—Plate 1, Fig. 28].

The following conclusions may be made as to the paleogeography of the Crimean-Caucasian region in the Upper Triassic. During the Norian and at the beginning of the Rhaetic a shallow warm sea existed at the marginal part of the Russian platform, along the modern northern slope of the Caucasian and Crimean mountains where calcareous sponges, hydroid polyps, corals and red and bluish-green algae developed in luxurious abundance,

Considering the morphology of the colonial organisms collected, the presence of calcareous algae, the abundance of calcareous crusts, the formation of which is probably connected with the activity of algae, a conclusion may be made as to an extremely shallow water basin where the accumulation of limestones proceeded almost at the surface of the water. During the Rhaetic an uplift of the sea-bottom took place, accompanied by temporary subsidences and a subsequent erosion. At the Crimea, at the beginning of the Lower Jurassic an accumulation of pebblestones and sands took place, the lower horizons of which may be referred to the Sinemurian stage. At the Caucasus the transgression is beginning evidently at a later period, towards the close of the Lower Lias. All these fluctuations of the sea are connected with the

Salghir mountain forming phase (2) (isolated by the present author) which, judging by recent observations, was very intense at the Crimea. Structurally Mountainous Crimea was closely connected with North Caucasus during the Triassic.

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HYDROBIOLOGY

CERTAIN NEW ELEMENTS IN THE COMPOSITION OF THE FAUNA OF THE BLACK SEA LIMANS AS AN EFFECT OF MARITIME NAVIGATION

By A. K. MAKAROV

(Communicated by S. A. Zernov, Member of the Academy, 12. IV. 1939)

There is quite a number of facts illustrating the part played by sea-going vessels in the spreading of hydrobionts in the various water reservoirs of the globe.

By attaching themselves to ships some of the hydrobionts may travel long distances finding their way to reservoirs where they may prove harmful. The Chinese crab known as the *Eriocheir chinensis* is just this kind of hydrobiont. It was introduced in Europe about 1912 and since then has been rapidly spreading in the waters of various countries approaching almost closely the waters of the USSR. Recently it was found near Leningrad, at Vyborg. The *Eriocheir*, as is well known, destroys the shore-line and is harmful to hydraulic constructions.

In the course of their study of the fauna in the northern limans of the Black Sea the University of Odessa and the Ukrainian Institute of Fisheries and Oceanography have come across facts which conclusively prove the part played by ships in the spreading of hydrobionts. The discovery in the Bug and Dnieper limans of the *Heteropanope tridentata* (Maitland) crab, hitherto considered to be an endemic of the Zuider-Zee in Holland, is undoubtedly of great interest.

The presence of this crab in the Bug and Dnieper limans was first noticed in 1936. The major studies of the fauna of these waters having been made in 1931, the crab must have made its first appearance in these limans between 1932 and 1935. In our opinion, the fact that in the Bug liman this crab is mainly haunting the Nikolaiev harbour and its vicinity shows that the *Heteropanope tridentata* had been introduced in these waters by ships coming from Holland to Nikolaiev. Here it is necessary to emphasize the statement made by Otto ([5]) to the effect that in April 1934 the *Heteropanope tridentata* made its appearance in large numbers in certain section of the Zuider-Zee, a case which had not been observed before. It is to be presumed that the fact noted by Otto stands in direct relation to the appearance of this crab in the Bug and Dnieper limans, since the first appearance of the *Heteropanope tridentata* in these limans roughly tallies with the date given by Otto.

We have investigated a few hundred specimens of the *Heteropanope tridentata* taken from the limans and in all essentials we found them answering the description given by Man (²) for this crab in the Zuider-Zee.

The *Heteropanope tridentata* is found in the Bug liman mainly in the section beginning with the Chervonoye village situated 3 km above the city of Nikolaiev and ending with the place where the Bug and Dnieper limans join. In the Dnieper liman this crab was found only once in the region of the Adjigiopol spit. From what the fishermen say, this crab is met with throughout the Bug liman up to the Balovnoye village and in the Dnieper liman at the Stanislav spit. The *Heteropanope tridentata* was first observed by the fishermen in the liman in 1936 and was nicknamed „the little spider".

Fig. 1. Crab *Heteropanope* (Maitland). Bug liman; April 1937 (× 2).

In the Bug liman the *Heteropanope tridentata* lives in waters of different salinity. On April 13, 1937, it was discovered in water containing 0.09‰ Cl, that is, practically in fresh water. In July and August, 1937, it was discovered in the liman in places where the water contained Cl between 0.909 and 2.27‰. In one instance it was found in water containing 4.261‰ Cl. Notwithstanding the definite euryhalinity of this crab, it has been found preferably to inhabit waters containing 2‰ Cl. The *Heteropanope tridentata* is found in waters with a content of oxygen varying between 3.84 and 8.21 cm³ per liter of water.

A drop in the temperature of water seems to have a depression effect on this crab. In April with the temperature of the water varying between 7.9 and 8.5° C, and in November with the temperature varying between 7.4 and 9.2° C, the crab has been found in a state of torpor; it has never been found in such a state in summer with the temperature of the water varying between 24.2 and 25.8° C.

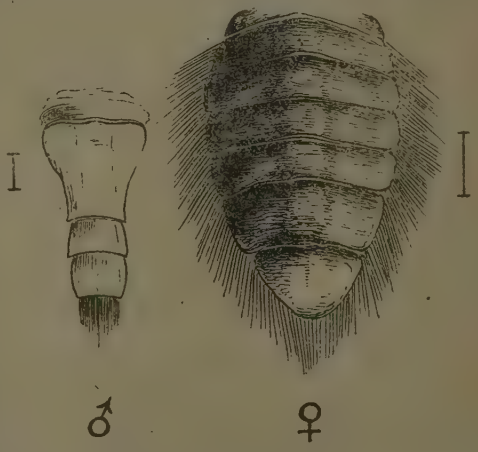

Fig. 2. Abdomens of *Heteropanope tridentata* (×5)

In summer the favourite haunts of our crab are those ports of the liman where the depth is between 2 and 4 m and the bed is covered with cockles or mussels. In the near shore part of the liman this crab has usually been found under some stones, but only in the vicinity of greater depths. In the winter the main mass of this crab probably moves to other parts of the liman with a slimy bed and depth of from 7 to 8 m. It was at such depths

that these crabs were found in large numbers in the months of April and November, 1937.

In the Bug liman this crab keeps to the biocoenosis of *Dreissena* which is typical of the Bug liman as distinct from the Dnieper liman. In the coastal zone, particularly in the region of the city of Nikolaiev, the *Heteropanope tridentata* has been found under stones amidst *Enteromorpha* sea-weeds and rose-coloured crusts of the *Melobasia* together with a large number of *Teodoxus* and *Cordylophora*. Sometimes the catch of the crab coincided with the schooling of numerous shoals of young *Mugilidae* in these waters. An enormous number of young *Heteropanope tridentata* varying from 1 to 4.5 mm in size was discovered in the bearded growth on the piles of the Varvarsky bridge. Here about 50 young crabs have been found on an area of about 20 cm² quite close to the surface amidst *Dreissena polymorpha*, *Cordylophora* and little *Corophiidae* tubes.

Symbols:
○—Place of occurrence of *Heteropanope*.
△—Same, according to information supplied by fishers.
×—Place of occurrence of *Zoea Heteropanope*, VII and VIII, 1937.

The female crab probably attains sexual maturity when about 12 months old. Man (²) found a female crab, 115 mm in size, with eggs. The female crabs with eggs found by us varied in size between 13 and 15 mm. In two instances the eggs were counted: one female 15 mm in size and weighing 1.40 g had 4300 eggs, the other, 13 mm in size weighing 1.10 g had 4800 eggs.

The *Heteropanope tridentata*, as a rule, probably spawns in June and July and sometimes at the end of May and in beginning of August. The *Zoea* answer the description given by Tesch (³) for the larvae of the *Heteropanope tridentata* in the Zuider-Zee. Twenty four *Zoea* among other fauna were caught with a Jeddy net of medium size at a depth of 7.9 m; this is the largest number of *Zoea* taken in one catch. The conditions in which the *Zoea* are found lead one to assume that the *Zoea* avoid the sweeter layers of water at the surface; they are more frequently found in the deeper layers of water with a content of Cl up to 6.96‰. In one instance *Zoea* was found in water containing 0.982‰ Cl.

From the observations made it is evident that the *Heteropanope tridentata* in the Bug liman attains the age of three years. By November of the first year of its existence the crab attains from 6 to 8 mm in width of carapace, some attaining even 10 mm. By April of the following year the females attain on the average from 10.5 to 11.5 mm in size and the males, from 14 to 17 mm. The larger specimens, between 17 and 20 mm in size, are probably three years of age.

Another example of the effect of sea-going vessels on the fauna of limans is the closed Hadjibey liman, near Odessa; this liman is divided from the sea by a thick bar. Until 1932, that is until the great inundation that followed the thawing of large quantities of snow in the basin of the liman, the waters of the latter were of considerable salinity. According to Zagorovsky ([1]), these waters in 1926 contained Cl from 31.042 to 36.825‰.

After the inundation the salinity of the Hadjibey liman was considerably reduced and rose again only in the last two or three years. In the autumn of 1938 and in February 1939 the content of Cl in the liman varied between 18.8 and 18.87‰ (determined by I. Kaliberdin). In the last few years some cutters were brought here from the sea for use in the waters of the liman. As a result a number of Black Sea varieties has spread in this liman. Since 1937 *Mytilaster lineatus* var., *Hydrobia ventrosa* (Mtg.), *Leander adspersus* Rathke and some others have been discovered in the liman.

Of particular interest is the appearance of the shrimp *Leander adspersus*. It has spread in the waters of the liman to an extent unknown in the bays of the Black Sea. Here the shrimp is undoubtedly of commercial interest. The occurrence of such enormous numbers of shrimps in the liman is to be explained by the absence of fish and other organisms feeding on them. In November 1938, in some parts of the liman the shoals of shrimps coming to the shore were so enormous that for a distance of 10 to 15 m from the shoreline they formed a compact mass, numbering close to 2000 shrimps with a total weight from 1.5 to 2 kg per m^2.

A study of samples caught on November 5, 1938, showed that the bulk of these shrimps were hatched in the same year; big specimens of the three year old age were totally absent; the biggest among the males as well as among the females attained 58 mm.

The study of the *Heteropanope tridentata* was carried out at the Zoological Institute of the Academy of the USSR by V. V. Makarov and myself.

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GENETICS

ROENTGENOMORPHOSES IN *DROSOPHILA MELANOGASTER* AS DEPENDENT ON TEMPERATURE OF DEVELOPMENT

By M. E. LOBASHOV and V. B. SOLODOVNIKOV

(Communicated by N. I. Vavilov, Member of the Academy, 13. IV. 1939)

The recently discovered possibility of producing simultaneously both phenocopies (1) or roentgenomorphoses (2) and mutations by a common external factor may be made use of in comparative study of the mechanism of action of external factors upon somatic and sex-cells. The autonomy characteristic of the latter cannot present any difficulties in distinguishing between, say, action of X-rays upon the sex or somatic cell; in the two cases the injury caused to the cell by X-rays tells on the former at the moment of its most intense activity. It is very likely that, when irradiated, processes and structures are the first to undergo alterations which at that moment play the leading part or determine the process of development of the cell. No matter whether we cause injury to the tissue producing organs or to the sex-germ, the injury is probably at the beginning en gros similar in cells of different sorts.

Starting from such general considerations, we may perhaps easier solve a number of questions relating to the nature of action of X-rays upon the mutation process by studying roentgenomorphoses. The purpose of the present communication is to elucidate the phenomenon of after-effect of X-rays in case of production of roentgenomorphoses in *D. melanogaster*.

Material and Method. Three normal lines of *Drosophila* were investigated; two of them („Samara" and „Bukhara") were recently (1936) taken from nature and introduced into culture; one line („Florida") comes from America and is long since bred under laboratory conditions.

Larvae of highest age (from 4 days to pupation) were irradiated by a dosage of 4000 *r*. Prior to irradiation the larvae were reared at a temperature of 25—27° C. Similarity of age was attained by allowing females to lay eggs during a short period (from 6 to 8 hours). After X-raying the whole lot of larvae was divided into several approximately equal groups; each group was then transferred to a different temperature for completion of its development, i. e. until the imagines emerged.

In order to attain maximum objectivity in registration of roentgenomorphoses we have chosen only one kind of morphose, notched wings (of the „notch" type) which are often met with when larvae of oldest age are X-rayed. Roentgenomorphoses were registered independently of the size of the notch or symmetry of its manifestation. They are summed up in the table; records

were taken, however, for females and males separately. The results are collected in the table.

Exper. No.	Origin of the lines of flies	$t°$ of development of the X-rayed flies in °C	Examination for morphoses, notches of the «notch»-type			X-ray dosage in r	Age of X-rayed larvae (in days)
			Total number or flies examined	Number of flies possessing notches of the «notch»-type	%		
1	«Samara»	15—17	663	40	6.0	4000	4
		25—27	1417	12	0.8		
2	«Samara»	15—17	343	44	12.8	4000	4½
		20	334	32	9.5		
		25	236	20	8.5		
		30	125	7	5.6		
3	«Samara»	15—17	317	166	52.3	4000	Prior to pupation
		25—27	498	84	16.8		
		15—17	289	14	1.2	Control without X-raying	
		25—27	1183	2	0.7		
4	«Bukhara»	15—17	314	125	39.5	4000	Prior to pupation
		25—27	470	19	4.0		
5	«Florida»	15—17	490	91	18.5	4000	Prior to pupation
		25	501	68	13.5		
		30	697	97	13.9		
6	«Florida»	25	202	19	9.4	4000	Prior to pupation
	«Samara»		231	46	19.9		

In case the action of X-rays upon the cell is exerted immediately at the moment of irradiation, a mere summation of distinct effects should be expected from application of an additional agent (temperature) after X-raying. As may be seen from the table, exper. 3, the results of our experiments have shown that low temperature (15—17° C) alone gives considerably less morphoses than in combination with the action of X-rays.

One should assume from the very fact of occurrence of morphoses („notch") that cells of the imaginal disk of the wing were at a sensitive period when X-raying was applied; the continuum of the process of development which produces the wing is most easily disturbed at this period. Therefore, if the effect of X-rays were dependent merely on disturbance produced at the moment of X-raying, one should anticipate that the change in temperature regimen of development would either increase or reduce the disturbance in the abnormally developing wing. Different temperatures should in this case be manifested in changes in the size of the notch, but no change in the number of individuals exhibiting morphoses should take place*.

As a matter of fact, however, we succeed not only in changing the size of the notch by temperature, but the number of individuals showing the

*It is possible that the notch, especially a small one, would be completely levelled in some of the individuals at high temperatures.

morphose, too (table). By means of an additional effect (temperature) upon irradiated imaginal disks of the wing it thus proves possible to establish the after-effect of X-rays. The results of experiment 1 may serve as especially strong evidence in favour of our interpretation of the after-effect of X-rays. In this experiment four day old larvae were X-rayed: they did not yet reach the period of sensitivity to formation of notches on wings. As a matter of fact, larvae irradiated with a dosage of 4000 *r*, at a temperature of 25—27°, display almost the same frequency of occurrence of individuals with morphoses (0.8%, exp. 1) as the control cultures at the same temperature (0.7%, exp. 3). At a lower temperature of development (15—17° C) the frequency of occurrence of individuals with morphoses attains in larvae of the same age under the same experimental conditions (1) 6.0%, i. e. is seven times greater than at high temperature.

One may thus arrive at a more general conclusion that, no matter at which stage (sensitive or non-sensitive) the tissue is treated with X-rays, it bears a kind of impression, or mark of X-raying, the manifestation of which depends on the conditions of development.

In connexion with the experiments presented above many questions arise, the elucidation and further detailed experimental study of which is needed. Let us dwell upon the concrete conclusions which follow from the present study.

1. The frequency of occurrence of roentgenomorphoses (notched wing of the „notch"-type) is directly connected with the temperature of development that follows X-raying. Lowering of the temperature of development (from 25—27° to 15—17° C) increases the frequency of occurrence of roentgenomorphoses (from 3 to 10 times), while an increase in temperature produces an opposite effect.

2. The velocity of physiological processes which determine the formation of the notch on the wing does not agree with the temperature coefficient Q_{10} (see exp. 4); the difference between the temperatures of development is 10°.

3. A comparison of frequencies of occurrence of morphoses in different lines has shown them to vary considerably in the ability of producing the morphose in question (exp. 6).

Besides, it may be noticed that sensitivity or, as we would call it, reaction ability of the „Florida"-line both to X-raying and to temperature changes, appears to be lower than that in the line „Samara". At the same dosage of X-rays „Florida" produces half the number of roentgenomorphoses (exp. 6) and responds but slightly to temperature changes by increasing or decreasing morphoses (exp. 5). In another way respond to X-rays and temperature changes the two other lines („Samara" and „Bukhara"). They give a higher proportion of roentgenomorphoses, and their response to temperature changes is considerably stronger. An impression is thus gained as if they possessed higher reactivity to changes in conditions than the „Florida"-line which has been bred for a long time under relatively constant laboratory conditions. We are inclined to ascribe to this fact a broader biological importance; under natural conditions one of the ways of selection leads to increase in reactivity in ontogenesis, i. e. to highest readiness of the organism to a quick reaction to changes in the surroundings. It is thus easy to conceive that lines of *Drosophila* taken from nature should possess higher reactivity than a laboratory line. Owing to selection after frequent alternation of generations under relatively constant conditions, loss of reactivity takes place.

However, since natural conditions do not uniformly vary as to their dynamics, forms taken from different habitats may differ in reactivity. We are able to observe this even in our experiments when two lines of *Drosophila* taken at the same time from nature are compared with regard to

their response to changes of temperature. One of them („Samara") was taken from the region of Kuibyshev, while the other („Bukhara") from the region of Bukhara. When temperature is changed by 10°, the difference in the rate of morphoses attains the tenfold (exp. 4), whereas in the line „Samara" it is only three times under the same conditions (exp. 3). From this index, as well as from many other physiological indices studied in the laboratory of A. P. Vladimirsky and constituting characteristics of these races, we may conclude that the „Bukhara"-line reacts stronger to changes in the environments than does the „Samara"-line.

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GENETICS

MIXTURE OF SPERM IN CASE OF POLYANDRY IN *DROSOPHILA MELANOGASTER*

By M. E. LOBASHOV

(Communicated by N. I. Vavilov, Member of the Academy, 13. IV. 1939)

While studying dynamics of germinal selection in males of *Drosophila* of different geographic origin, we succeeded in establishing that their spermatozoa differ in viability or physiological activity. It appeared of interest to find out what is the behaviour of these spermatozoa when they are mixed in the receptaculum of the female.

A number of investigations on *Drosophila* ([1], [2], [3]), *Ortoptera* ([4]) and on the silkworm ([5]) have shown that complete or partial mixture of sperm takes place in cases of polyandry. In conflict with the evidence available is the early work by Dubinin ([6]), the author stating in a rather categorical form that „a new portion of spermatozoa entering receptacula completely eliminated the old one which is never manifested again" (p. 150).

Almost all the authors who investigated the question we are interested in failed to pay due attention to the comparative study of viability of sperms of different genetic constitution. Only Nachtsheim ([3]) pointed out in his short summary that longevity of sperms belonging to males of different genetic constitution varies both in different species and within a single species *(D. obscura)*. It is obvious that when different males copulate with one and the same female the number of the progeny produced by the latter from the one or the other male would in the first line depend on the amount of sperm ejaculated; however, the viability of sperms is also one of the decisive factors.

In this short communication we are going to touch upon two basic problems: first, is there mixture of the sperm in case of repeated copulations; secondly, does elimination of sperm occur when sperms of different constitution or different origin are mixed together; and if so, to what extent.

M e t h o d. In all the experiments conducted at the same time males of different constitution were about of the same age. Prior to crossing both the males and the females were kept out of copulation during several days. Copulation of the female with the first male was controlled, no such control being applied to the mating with the second male. After copulation the first male was removed and replaced by another one for a period of time not over 20 hours. Then the females were shifted to fresh nutritive medium every two days until they ceased to lay fertilized eggs. The progeny of females which have copulated twice was treated separately according to the date of egg deposition.

Exper No.	Conditions of experiment	Scheme of doudle cross and males' origin	Index	Date of shifting ♀♀ after repeated copulation: 1* Progeny from 1 ♂	1* Progeny from 2 ♂	2—4 Progeny from 1 ♂	2—4 Progeny from 2 ♂	4—6 Progeny from 1 ♂	4—6 Progeny from 2 ♂	6—8 Progeny from 1 ♂	6—8 Progeny from 2 ♂	8—10—12 Progeny from 1 ♂	8—10—12 Progeny from 2 ♂	Total progeny from 1 ♂	Total progeny from 2 ♂
1	Normal crossing	♀ *sc v f car* / *sc v f car*	Quant.	252	47	366	344	214	206	78	128	34	71	944	796
		× 1 ♂ «Samara» × 2 ♂ *sc v f car*	%	84.4	15.6	51.6	48.4	50.9	49.1	37.8	62.2	32.3	67.7	54.2	45.8
2	» »	× 1 ♂ «Bukhara»	Quant.	202	46	425	624	325	584	100	171	17	69	1 039	1 494
		× 2 ♂ *sc v f car*	%	81.5	18.5	40.5	59.5	35.7	64.3	36.7	63.3	19.7	80.3	41.7	58.3
3	» »	× 1 ♂ «Florida»	Quant.	257	103	147	128	74	131	2	9	—	—	480	376
		× 2 ♂ *sc v f car*	%	70.7	29.3	53.5	46.5	36.1	63.9	18.1	81.9	—	—	56.4	43.6
4	» »	× 1 ♂ *sc v f car*	Quant.	83	539	57	452	19	77	—	—	—	—	159	1 038
		× 2 ♂ «Florida»	%	13.3	86.7	11.2	88.8	19.7	80.3	—	—	—	—	13.0	87.0
5	The 1st ♂♂ were X-rayed before copulation	× 1 ♂ «Samara»	Quant.	445	382	650	2 239	436	1 769	50	974	7	103	1 588	5 472
		× 2 ♂ *sc v f car*	%	53.8	46.2	22.4	77.6	19.7	80.3	4.8	95.2	6.0	94.0	22.4	77.6
		× 1 ♂ «Bukhara»	Quant.	617	227	507	2 509	441	2 364	99	854	15	77	1 679	6 031
		× 2 ♂ *sc v f car*	%	73.1	26.9	16.4	83.6	15.7	84.3	10.4	89.6	16.3	83.6	21.7	78.3
6	♀♀ were centrifuged after repeated copulation	♀ *yy* × 1 ♂ «Bukhara»	Quant.	—	—	95	244	71	190	13	140	0	28	179	602
		× 2 ♂ white	%	—	—	28.0	72.0	27.2	72.8	8.5	91.5	0	100	22.9	77.1
	Control	× 1 ♂ «Bukhara»	Quant.	—	—	236	345	121	303	85	201	22	24	464	873
		× 2 ♂ white	%	—	—	40.6	59.4	28.5	71.5	29.6	70.4	47.8	52.2	34.7	65.3
		× 1 ♂ white	Quant.	—	—	63	569	27	203	4	46	—	18	94	841
		× 2 ♂ «Bukhara»	%	—	—	9.9	90.1	11.4	88.6	8.0	92.0	—	100	10.0	90.0

* Progeny from the fertilized eggs which were deposited by females in the period between the first copulation prior to their shifting to the fresh nutritive medium after repeated copulation.

Crossing was made according to the following two schemes: 1) ♀ $\frac{sc\ v\ f\ car}{sc\ v\ f\ car}$ × 1st ♂ «Samara», 2nd ♂ *sc v f car*, the progeny being identified by the phenotype of the females in F_1; 2) ♀ *yy* × 1st ♂ «Bukhara», 2nd ♂ white (or in the reverse direction), in which case the progeny was classified by the phenotype of the males in F_1. In all the experiments all the progeny of the twice mated females was recorded, each experiment involved no less than 20 such females, their number often being much higher than this. Aside from ordinary crosses, as just described, we have conducted experiments with X-raying sperm of the first male; an experiment with centrifuging two times mated females was also conducted.

The results of the experiments are summarized in the table.

These experiments allow one to make the following short statements: 1) after the double copulation complete mixture of the sperm takes place in the receptacula of the female (experiments 1, 2, 3 and control to exp. 6); 2) dynamics of the number of the progeny of the 1st and 2nd male depends on the quantity of sperm introduced by the one or the other male into the receptacula of the female as well as on the viability of the sperm, the latter in its turn being dependent upon the genetic constitution of the male (compare exp. 4 with exp. 3 and the control to exp. 6); 3) X-raying males prior to mating (dosage 4000 *r*) lowers the viability of the sperm (part of the sperms may possibly be destroyed by the action of X-rays); one may judge of this by the change in numerical relation of the progeny of an X-rayed male to that of a 2nd one (compare exp. 5 with exp. 1 and 2); 4) centrifuging (2000 revolutions per minute, exposure 10 min.) lowers the viability of the sperm of the first male, like X-rays do (exp. 6).

Some General Remarks. The prevalence of the progeny of the one or the other male in double crosses may be explained by quantitative predominance of the sperm of the one male, as it was done by Beliaiev [5] in his work on silkworm. However, if this were the only factor and no elimination of sperms was taking place, we should on the whole obtain equal relation between the progenies of the two males in our experiments on «direct» and «reciprocal» crosses (exp. 4 and 3 and control to exp. 6). It was not observed in our experiments. In the cross ♀ $\frac{sc\ v\ f\ car}{sc\ v\ f\ car}$ × 1st ♂ «Florida», 2nd ♂ *sc v f car*, there occurred 56.4% of progeny of the 1st ♂ and 43.6% of the 2nd ♂; in the reciprocal cross there occurred only 13.0% of the ♂ *sc v f car*. The same phenomenon was observed in the cross: ♀ *yy* × 1st ♂ «Bukhara», 2nd ♂ white, where 34.7% of the progeny belongs to the 1st ♂ and 65.3% to the 2nd ♂; in the reciprocal cross only 10.0% of the progeny is produced by the ♂ white. These facts convinced us that elimination of sperms takes place and that mortality among sperms of the males *sc v f car* and white is considerably higher than in the normal lines. Thus, hereditary and physiological constitutions of the sperm (the latter being determined by the former) prove to be one of the decisive factors which control the behaviour of sperms in mixed state. To account for elimination of the sperm of the 1st male Dubinin [6] has advanced [under the influence of Godlewsky's work [7]] the hypothesis of biochemical antagonism between the «old» and the «new» sperm. If there exists such an antagonistic influence of the sperms upon each other (one should take into account the secretion of the sexual glands, too), it may be characteristic of widely remote taxonomic groups (for instance, in Godlewsky's experiments on mixed sperms of round worms with that of echinoderms) and is hardly probable in the case of sperm mixture of males belonging to the same species. Polyandry within a species appears

to be an adaptation which is of high advantage for preservation of the species; to postulate biochemical antagonism between sperms within a species as a general law means to reject the expedience of polyandry. Very advantageous conditions for the existence of the species are created by polyandry, e. g., increased intensity of selective fertilization, wider scope of combinatory variation, etc.

From our point of view, elimination of sperms of the one or the other constitution depends on their ability of competition in the process of fertilization, while the intensity of elimination depends on the strength of selection. Therefore, sperm of low viability is used to a greater extent (as may be judged from the progeny obtained) in single copulation, i. e. females lay fertilized eggs during a longer period of time; it is used much less in double copulation.

More than 30 sperms may penetrate into a single egg of *Drosophila*, while the «honour» to fertilize the egg-cell fall to the lot of but one of them; besides, if it be taken into account that an increase in concentration of the sperms results in an increase of polysperms ([8]), our conclusion that dying of the sperms occurs during the process of fertilization through selective fertilization appears still more convincing.

We cannot analyze here in detail all experiments and results of our predecessors; we shall only point out that they all seem to corroborate our general conclusion on the existence of selective fertilization in animals.

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GENETICS

MUTATION RATE IN IMMATURE SEX-CELLS OF *D. MELANOGASTER*, ITS DEPENDENCE ON X-RAY DOSAGE AND METHOD OF ITS DETERMINATION

By W. I. GRATZIANSKY, V. A. GRIGORJEV and V. G. KARABANOV

(Communicated by N. I. Vavilov, Member of the Academy, 13. IV. 1939)

The rate of mutations produced by X-raying grows in direct proportionality to the X-ray dosage; this fact, however, is a cummulative effect of heterogeneous processes. Aside from the direct proportionality of transgenations, there is a more rapid increase in that sort of mutations which are connected with double breaks in chromosomes (translocations, inversions, etc.); proportionality of their increase to the X-ray dosage appears to be a parabolic function with its exponent (x) equal to 2 or $2>x>1$ ([1], [2], [6], [7]); on the other hand, it was shown by Lobashov ([3], [4], [5]) that there exists germinal selection which eliminates a fraction of the mutations produced.

In considering determination of the frequency of mutations produced by various dosages of X-rays in immature sex-cells we should the more expect the effect of germinal selection. We think that X-raying immature sex-cells should be of twofold importance (to a still greater extent than in case of mature gametes): increase of the X-ray dosage enhances the mutation process and, besides, renders germinal selection more intense. Study of the dependence of the mutation rate in immature sex-cells on the X-ray dosage is, therefore, one of the central points in the solution of the question as to the character of the mutageneous action of X-rays. Very likely it is here that one should expect disproof of the idea of the «direct» physical action of X-rays upon the gene and, instead, proofs in favour of the cyto-physiological mechanism of mutation. In accepting the latter point of view we are seeking to add to evidence produced by M. E. Lobashov ([5]). The present paper is a part of a series of works on the physiology of the mutation process conducted in the laboratory of genetics and experimental zoology of the Leningrad State University; however, it has an additional purpose, viz., to demonstrate the possibility of comparing mutation rate in mature and immature sex-cells which is disputed by Shapiro ([8]). Demonstration of such a possibility is of great importance; should it be otherwise, it were impossible to estimate quantitatively the relation between the mutation process and germinal selection and, consequently, to understand fully the biological nature of mutation.

Material and Method. From 96 to 108 hour old larvae were subjected to irradiation; they all belonged to the «Samara» line and the

same installation was applied throughout the experiment. X-rayed males were crossed with females $\frac{ClB}{sc\ v\ f\ car}$ in order to record all sex-linked mutations. Cases of non-disjunction of the X-chromosomes were not included. F_2 cultures which contained less than 30 flies were not recorded either. Only possible cases of crossing over in the vermilion region in *ClB* females were thus not taken into account.

The results of the experiments are summarized in the table. Extremely low proportion of mutants in all the experiments and statistically insignificant mutageneous effect of extreme dosages of X-rays (200—6000 *r*) stand out.

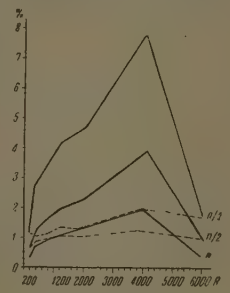
Dosage of irradiation in *r*	Number of cultures	Number of lethals	$q\% \pm m_q$	Diff $\pm m_{diff}$
200	331	1	0.302 ± 0.301	0.386 ± 0.437
400	727	5	0.688 ± 0.306	0.200 ± 0.499
800	674	6	0.888 ± 0.361	0.172 ± 0.577
1200	566	6	1.060 ± 0.427	0.247 ± 0.649
2000	917	11	1.307 ± 0.375	0.693 ± 0.580
4000	1000	20	2.00 ± 0.436	1.562 ± 0.619
6000	230	1	0.435 ± 0.433	

Further discussion of the results obtained requires certain specifications relating to the methods of statistical evaluation of the proportion of mutations in experiments with immature sex-cells. As quite soundly pointed out by Shapiro ([8]), application of the usual formula $m_q = \pm\sqrt{\frac{p(100-p)}{n}}$ to calculate the average error of the mutation per cent would be incorrect in case of immature cells. It appears to us that, owing to the same reason, the per cent of mutation should not be calculated itself from the number (n) of chromosomes analyzed in F_2: both in case of $q\%$ and m_q the usual methods of calculation would involve the number of chromosomes investigated (n) which is known to be greater than the actual number of irradiated chromosomes.

It does not mean, however, that mutation rates in mature and in immature sex-cells cannot be compared with each other. Starting from the number of mutations found for a certain value of n, one may calculate the per cent of mutations for cases of their production one, two or three, etc., cell divisions prior to the completion of gametogenesis ; to do this one should express the amount of mutations observed in per cent of $n/2$, $n/4$, etc. (instead of n) according to the number of mitoses, between the moment of irradiation and completion of spermiogenesis. It is understood that there are no «bunches» among the mutations found. Taking these values as a kind of theoretical maximum we may attribute shortage in mutations to elimination through germinal selection. Even when «after-effect of X-rays» (which is not proved yet) is admitted, the situation remains unchanged: the greater percentage of mutations should be expected in this case, the longer period of maturation is available for the after-effect. In other words, the theoretical frequency of mutations, on the one hand, and the rôle played by germinal selection, on the other, should be expected to increase inasmuch as the departure of the actual values of $q\%$ from the expected ones should rise. (We emphasize the difference between mutation percentages at 4000—6000 *r*: it is most significant which serves as direct evidence in favour of the twofold rôle of X-rays in determination of the mutation rate.) On the other end of the scale is the maximum possible identity of mutations: of every two mutations

at $n/2$, of every four mutations at $n/4$, etc., such calculation supplied the other theoretical series of minimum frequency of mutations at different stages of gametogenesis. Actual frequencies of occurrence of mutation should evidently vary from experiment to experiment (depending on the intensity of germinal selection), but not beyond the values of the two series. Besides, the more the intensity of germinal selection, the less identical mutations would survive till fertilization, i. e. the more would the experimental values approach the corresponding theoretical series of frequencies (for $n/2$, $n/4$, etc.).

With reference to our material such calculations were made in the following way. According to unpublished data of A. I. Zuitin, 96—108 hour old larvae should pass through the two meiotic divisions, meiosis being first initiated in them at this age at 25°. In our experiments larvae of this age were X-rayed. One may think, therefore, that experimental conditions demanded calculation for two divisions (i. e. calculation of percentage of $n/4$). The diagram (figure) illustrates the results of calculations according to the above method: 1) graph of the frequencies empirically found (n), 2) graph of theoretical maximum frequencies at $n/2$ and $n/4$, and 3) graphs of theoretical minimum frequencies at $n/2$ and $n/4$.



As shown by the graphs, calculations do not change the observed relation qualitatively; however, comparison between the extreme possibilities (solid and dotted graphs for $n/2$ and $n/4$) reveal sharp distinction between them. The latter is expressed in the fact that dotted graphs (series of minimum theoretical mutation percentages, on the assumption that they are to the limit identical to each other) do not exhibit statistically significant decrease of frequency of lethals in the interval from 4000 to 6000 r. In corresponding solid graphs (which express the maximum frequencies of mutations assuming that there are no identical at all) this decrease is well expressed and is exaggerated by calculation for $n/2$ and $n/4$. This means that, assuming maximum identity of mutations in individual experiments, we think that all the occurring mutations multiplied in the same tempo as the non-mutated cells, i. e. that no germinal selection took place. Inversely, assuming complete non-identity of the mutations found we postulate the action of germinal selection. The difference between the dotted and solid graphs should be attributed to germinal selection both for $n/2$ and $n/4$.

On the basis of the communicated above a concrete way may be indicated for estimating intensity of germinal selection and its rôle in determination of the percentage of mutations which arise when immature sex-cells are X-rayed. This method consists in comparison of maximum and minimum series (as offered by us) for $n/2$, $n/4$, etc. with the proportions of non-identical mutations actually found in experiment. Experiments corresponding to each other in the number of actually irradiated chromosomes $n/2$, $n/4$ may thus demonstrate one of the factors which determines the integral dependence of the mutation frequency on the X-ray dosage. Simultaneous arrangement of the percentages of chromosomal reorganizations among the mutations found should help in determination of the other constituent—the increase in the exponent of this functional dependence. Such investigation is planned by us for the next future.

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GENETICS

DIFFERENT VIABILITY AMONG FLIES OF *DROSOPHILA MELANOGASTER* HETEROZYGOUS FOR LETHALS

By R. A. MASING

(Communicated by N. I. Vavilov, Member of the Academy, 12. IV. 1939)

By our previous investigations (1) was established the interesting fact that flies of *Drosophila melanogaster* heterozygous for lethal mutations in the II chromosome exhibit higher viability* than those bearing no lethals. In the paper cited above we have advanced three hypotheses to account for the phenomenon observed, viz: 1) the lethal gene in question might be in itself a dominant gene for increased viability possessing recessive lethal effect; 2) the lethal gene at the moment of its origin has a deleterious effect upon the life of the individual, the increased viability observed under laboratory conditions being the result of interaction between the lethal and the dominant modifiers accumulated in the course of evolution and neutralizing the action of the lethal; 3) the lethal gene studied is by mere chance linked with a dominant gene for increased viability.

To decide upon the correctness of the hypothesis advanced we undertook comparative study of the viability of flies heterozygous for individual regions of the chromosomes investigated, both containing a lethal and lacking it.

As already pointed in the preceding paper, the third hypothesis appears to us hardly probable. While inspecting for lethals 150 chromosomes of the Crimean population and 170 chromosomes irradiated by X-rays, we were especially looking for genes of increased viability, but failed to find even a single case of such sort.

Lethals were localized with the aid of the $\frac{Cy}{BlL^2}$ line; three lethals most conveniently located were chosen for analysis. Two of them (*T*201 and 627) arose under natural conditions and one (15*M*7)—under the influence of chemicals (it was handed over to us by V. V. Sakharov to whom thanks are due).

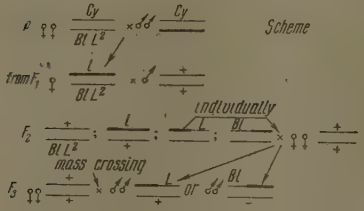
The following crosses were arranged in order to obtain crossover males bearing lethals and lacking them (see diagram).

As may be seen from the diagram, we have obtained in F_2 the following 4 phenotypes: 1) normal, 2) BlL^2, 3) Bl and 4) L^2.

Crossover males (there occurs no crossing over in males) were individually

* The term viability is here used to denote the comparative viability in the course of development, not the vitality of adult flies, the study of the latter being only begun by me.

crossed with normal females*. From crossing males $\frac{Bl}{+}$ with normal females both Bl and normal flies were obtained; from crossing males $\frac{L^2}{+}$ with normal females both L^2 and normal flies were obtained. In both cases, when specific effect of the chromosome region investigated is excluded, the expected ratio is 1 : 1. In order to obtain larger number of flies and to calculate the ratios in question, crossover males Bl and L^2 obtained from individual crosses of males Bl and L^2 with normal females, were crossed with normal females, this time in mass cultures (see diagram). The majority of crossover males were checked for presence, resp. absence, of the lethal by crossing them with females Cy/l of the line investigated. When lethal is present, no flies L^2 non-Cy arise from the cross $Cy/l \times lL^2/+$, the same being true of crossover males Bl; in the presence of the lethal no Bl non-Cy flies are obtained.



For the sake of control all crosses of the scheme were made, males $\frac{Cy}{+}$ were taken instead of Cy/l males.

In the course of these crosses we were able to verify once more the ability of the chromosome studied to increase vitality of the flies heterozygous for it. In F_2 more normal flies than BlL^2 flies should be present among non-crossovers, if the presence of the chromosome in question increases viability. In Table 1 are presented the results of counts of the normal flies and of the flies BlL^2 in F_2. Statistically significant preponderance of normal flies is observed in all cases.

Table 1

$$♀ \frac{l}{BlL^2} \times ♂ \frac{+}{+}$$

Culture No.	Observed % of normal flies	Expected % of normal flies	$\frac{\text{Diff}}{m}$	n
627	54.6	50 ± 1.46	3.1	1 184
T 201	63.8	50 ± 2.00	6.9	625
15 M 7	56.8	50 ± 1.51	4.5	1 092
Control	51.8	50 ± 1.05	1.7	2 244

Lethal 627. It is located to the right of L^2; therefore, crossover flies Bl will possess the lethal, whereas crossovers L^2 should lack it (see diagram). Preponderance of normal flies over BlL^2, albeit a slight one, is observed in F_2. Altogether were studied 5 crossover males L^2 (2 of them were checked for the absence of the lethal) and 5 crossover males Bl (three of them checked for the presence of the lethal). In all cases in the progeny of crossover Bl flies of the phenotype Bl were predominating, and in the progeny of crossover males L^2 the observed ratio of L^2 flies to normal ones coincided

* The normal line used was constantly perpetuated by strict inbreeding. Fine individual cultures with virgin flies were started in each generation; flies were taken from the best vial for perpetuation of the line. Selection for good viability was thus practiced all the time.

with the expected ratio (1 : 1). In Table 2 are summarized the data both for the progeny of the L^2 males and of the Bl males.

Table 2

Culture No.		♀ $\frac{+}{+}$ × ♂ $\frac{L^2}{+}$				♀ $\frac{+}{+}$ × ♂ $\frac{Bl}{+}$			
		Observed % of L^2 flies	Expected % of L^2 flies	Diff / m	n	Observed % of Bl flies	Expected % of Bl flies	Diff / m	n
627		51.8	50 ± 1.06	1.6	2 222	62.2	50 ± 1.05	11.6	2 112
*T*201	No. 14 Nos. 211.15	58.8 51.3	50 ± 2.25 50 ± 0.84	3.9 1.5	493 3 553	51.0	50 ± 0.86	1.2	2 920
15*M*7	with lethal without lethal	59.8 49.0	50 ± 1.84 50 ± 1.59	5.3 0.6	736 992	53.4	50 ± 1.97	1.7	646
Control		51.2	50 ± 0.90	1.3	1 746	51.2	50 ± 1.20	1.0	1 746

We may suppose on the basis of these data that the lethal gene is a gene of increased viability but lethal in homozygous condition (1st hypothesis, see above), or that the dominant gene increasing viability is accidentally linked with a lethal one (3rd hypothesis of little probability), the linkage being strong. The presence of a modifier (2nd hypothesis) may be also suggested, the modifier being again close to the lethal.

Lethal *T*201. It is located in the region *Bl*; therefore, crossovers *Bl* should lack the lethal, and crossovers L^2 should possess it. In F_2 (Table 1) we witness a very strong preponderance of normal flies over *BlL*. Quite another aspect is gained from the analysis of the progeny of crossover males.

Four *Bl* males and four L^2 males were investigated (the latter checked for the presence of the lethal). The progeny of the *Bl* males, as should be expected, gave normal ratios in all cultures; the pertaining data are summarized in Table 2. Among 4 L^2 males only one (No. 14) produced progeny with the expected preponderance of L^2 flies over the normal ones; as shown by analysis for *Cy*/*l*, this particular male possessed the lethal. In the progeny of the remaining three males (Nos. 2, 11 and 15) the preponderance of the L^2 flies over the normal ones was statistically insignificant (Table 2), although, as shown by analysis, they had the lethal. It should be reminded that the chromosome in question gave in F_2 (Table 1) statistically entirely significant preponderance of normal flies over BlL^2 flies. When individual regions of chromosomes containing lethal are studied, different results are found in progenies of different males. This makes one to suppose (in our opinion on a rather sound basis, too) that the increased viability is caused here by cooperation of the lethal and the modifier (2nd hypothesis), which is located so far from the lethal that crossing over between them often takes place.

Lethal 15*M*7. It is located in the region *Bl*. Crossovers *Bl* lack the lethal, crossovers L^2 bear the lethal. Three males *Bl* and 9 males L^2 were investigated (the two types were checked for the absence and presence of the lethal, respectively).

In F_2 we find again a verification of the augmented viability of the flies heterozygous for the chromosome studied (Table 1). Crossover males *Bl*

lacking the lethal produced normal ratio of *Bl* flies to normal ones (1:1). The results are summarized in Table 2.

In the progeny of crossover males L^2 different results were obtained depending on whether or not males L^2 possessed the lethal. Five males lacking the lethal gave normal ratio (1:1), whereas 4 males possessing the lethal produced preponderance of L^2 flies over the normal ones (Table 2).

These results make one to conclude that the 1st or the 2nd hypothesis are more probable in the case of the culture in question, viz., either the lethal is itself a dominant gene for increased viability producing lethal effect in the homozygote, or it is so closely linked with a gene for increased viability (or with a modifier) that the latter fails to increase unless the lethal is present.

In the given line $15M7$ in the chromosome studied there is a recessive gene *bw* located considerably to the right of L^2. It could be supposed that its presence causes increased viability, i. e. that such is its pleiotropic effect [such a phenomenon we had observed in the case of gene *ebony* in the III chromosome, see Masing (1)]. Our analysis shows that the gene *bw* produces no effect whatever upon viability. Flies L^2 possessing the gene *bw*, but lacking the lethal, show no increased viability.

Study of individual regions of chromosomes in all the lines heterozygous for the lethals has thus demonstrated a connexion between the observed increase in viability and the presence of a lethal; this connexion is, besides, different in different lines.

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PLANT PHYSIOLOGY

ACCELERATION OF DEVELOPMENT AND FRUITING OF FRUIT-TREE SEEDLINGS

By J. POTAPENKO

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Late entering of fruit-tree seedlings into fruiting stage draws out the breeding process considerably and it is also after a long time has passed that the results of a biological experiment on those plants can come into light. Naturally enough, the acceleration of their development has been endeavoured by many investigators.

I. V. Michurin ([2],[3]) has emphasized the great importance of development acceleration in plants as a means to speed up the breeding process as well as to change the nature of vegetation period of fruit-trees on their colonization in more northern regions.

A review of literary data on the subject shows that the various means applied by different authors to induce early fruiting as: girdling, clipping of roots, grafting upon a dwarf stock, raising in pots, yearly transplantation and even insertion of seedling cuttings into the top of fruit-trees, all failed their intended purpose [experiments of Crane ([7]), Spinks ([8]) and Turasse ([4]) with seedlings of plum, apple and pear respectively, the latter made as early as in the eighties of last century]. The rôle of grafting in acceleration of fruiting in seedlings has been discussed by Michurin ([2]).

The analysis of these experiments shows that their authors paid special attention to nutritional conditions where they expected to find the chief responsible cause of late fruiting. Important of course is the rôle of these conditions in fruiting, but it does not exhaust the list of factors determining the development of fruit-tree seedlings. And it is by having taken no account of this fact that their failure to induce early fruiting should be accounted for.

At present the possibility to accelerate the development of plants is proved by the works of Lyssenko ([1]). The theoretical statements of Michurin and Lyssenko have built the basis from which the attempts at solving the problem with reference to perennial seed plants can start.

The experiments carried out on this foundation ([5],[6]) have led us to the conclusion that the late fruiting of fruit-tree seedlings is their inherent quality and that the possibility of accelerated development should be sought in the nature of the plant concerned.

The yearly cycle of plant's development is made up of different periods or phases. For each phase to attain its maximum speed definite conditions must be available. The transition from one phase to another is effected at once

by way of changed requirements to developmental conditions. Not only is this course of development an adaptation of the plant to the change in conditions throughout the year, but it is also its response to climatic fluctuations in different years. One who knows what conditions of temperature, light, etc. ensure the most rapid course of the yearly developmental cycle, is able to accelerate development and fruiting by giving a certain number of respective cycles to the seedlings.

Our experimental work along this line has been run with the seedlings of apple, cherry and vine. It has involved hybrids as well as seedlings of free pollination. Considerable number of seedlings were propagated vegetatively. Part of the vegetative offspring served as control, while the rest was given conditions for accelerated development. These are the results obtained in the experiment on cherry. Two year old seedlings, variety «Ideal», free pollination, grown from the seed planted in beds were in the spring of 1935 pricked

Fig. 1. Cherry seedling, variety «Ideal», toward the end of the second year. Left, control No. 48, passed two vegetations. Right, seedling No. 75, has produced flowers during the fourth vegetation.

Fib. 2. Cherry seedlings, variety «Ideal», toward the end of the second year. Left, control No. 24, passed two vegetations. Right, seedling No. 90, has produced flowers during the fourth vegetation.

off either in the open ground (276 plants) or (30 plants) into clay pots of small size (bright and upper diameter, 22 cm). The amount of soil per pot was measured by weight. Potted seedlings were divided into 4 series. 7 plants, 7 plants, 8 plants and 8 plants were given normal, continuous (sunlight plus electric light), 12-hour and 8-hour day lengths respectively. After the end of the vegetation in the autumn of 1935 the seedlings were subjected to the action of low temperature (from +2 to +7° C) for a period of 2—2.5 months. In December of 1935 after being transferred to the hothouse they started growing and two of them produced flowers. One of the latter had been given normal day length, the other, continuous illumination. In 12-hour and 8-hour series there were no flowering seedlings.

Of 276 seedlings pricked off in the garden plot only 4 (i. e. 1.5%) produced flowers in the spring of 1936, i. e. on fourth vegetation. Further experiments carried on with the seedlings of cherry and apple from Central Russia and of more northern origin have shown those plants to complete vegetation

earlier on normal day length (at Michurinsk it is a long day which becomes a short one toward the end of vegetation). On continuous illumination the seedlings make their first growth very quickly, but their transition to dormancy is slowed down.

With short day lengths (12 and 8 hours) both growth and development are greatly delayed and fruit bud primordia fail to form.

The seedlings which are ready to bear fruit can, when given long or continuous days, form buds and produce flowers and fruits even if the cycle of development is curtailed. Thus, the seedlings which make two yearly cycles within one calendar year are able after having started the second vegetation on July 1 to complete it before the advent of autumn frost (under conditions prevailing at Michurinsk) and are forming fruit bud primordia being given additional electrical light. After having passed through the phase of low temperature in autumn and early winter the seedlings, transferred to the hothouse toward the end of December, start vegetation and flowering, i. e. make the yearly cycle of development in 6 months ([6]). In juvenile age, however, they fail to form fruit bud primordia, though they may have been given long or even continuous days throughout summer. We see therefore that if light is a very potent and at certain stages even a crucial factor in plants development, nevertheless, the most favourable conditions of light will not suffice alone to induce accelerated development in seedlings. In order to accelerate development we must give the plant the requisite number of developmental cycles which correspond to an equal number of years pressed into shorter intervals.

February 17, 1937, seeds of cherry «Ideal» were sown in a box in the hothouse. Seedlings appeared toward the end of the month. April 3 they were pricked off into clay pots. From the middle of April till the onset of autumn frosts they were grown out-of-doors. In the autumn of 1937 they were divided into two series, 39 plants each. The series were equivalent with regard to growth strength and morphological peculiarities of their seedlings. The plants of one of the series were left in the open to serve as control. The others were treated with low temperature ranging between +1 and +6° C and on December 27, 1937, they were pricked off into pots 31 cm high and just as broad in the upper part. The potted plants were transferred to the hothouse to pass the second vegetation. Control seedlings were also transplanted into pots of the same size in April 1938 to pass the second vegetation in the open until the advent of autumn frosts. The seedlings in the hothouse completed growth April 1, 1938, and on April 19 they were brought into a cold room ranging in temperature between +3 and +5° C. June 29 they were placed out-of-doors and July 1 began the third vegetation. Since in July the natural days grow shorter, the seedlings were given weak electric light in the night, on the average 50 candles per m^2. Added light was supplied from July 1 till the end of September, 1938. In the second half of October the leaves in the experimental seedlings took on autumn colour and many of them fall. At that time the leaf fall was taking place in control plants too. From the second half of October until 2—2.5 months later the experimental plants were passing low temperature phase between +1 and +6° C. Then they were transferred to the hothouse for fourth vegetation. On January 1939 the seedlings came into growth and ten of them (25.6%) produced flowers. At the time when the controls started the third vegetation, the hastened seedlings had completed the fourth which ended in flowering. Among the controls not a single flowering plant was observed as late as in the third year, and it will be obvious that not until the fourth year has come they are able to flower.

By the end of the second year counted from germination, we recorded the condition of seedling with reference to height, shoot elongation and stem

diameter. Toward that time the control plants had made two vegetative cycles, while the experimental ones had completed the fourth vegetation.

However, the growth intensity in itself was not the cause of the early fruiting, for in the open ground seedlings of more vigourous growth will not fruit until after they have attained a definite age, while small plants grown in small vessels can form fruit bud primordia if they are ready to enter this stage. Potting is also of little importance, since the controls grown in pots of equal size did not form fruit bud primordia. The crucial point in development acceleration and early fruiting is to make plants pass more quickly the historically determined cycles during which their internal condition is reiteratedly changing through their interaction with the environment.

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PLANT PHYSIOLOGY

HIBERNATION AND VERNALIZATION OF *LALLEMANTIA IBERICA* F. et M.

By D. J. VAKULIN

(Communicated by N. I. Vavilov, Member of the Academy, 13. IV. 1939)

Literary data on the hibernation of *Lallemantia* are very meagre. It is known, for instance, from the experiments of the Krasnodar Agricultural Institute that the seeds of this plant sown late in autumn, 1935, have produced seedlings as early as in December on the advent of thawing. But in spite of low temperature in the winter (—9.8° C without snow cover and — 19.4° C with a snow cover 15 cm in thickness) nearly 50% of plants have survived.

These data refer probably to the white-flowering form of *Lallemantia*. Respecting the blue-flowering form, however, no data are available in the literature. We were the first, perhaps, to follow up its variation in response to the winter cold and are able now to draw some conclusions.

On a plot where in 1937 vernalization experiments with *Lallemantia* had been run we observed young plants to have appeared spontaneously in the spring of 1938. These proved later to belong to the blue-flowering form of *Lallemantia* No. 90. The plant had survived winter and on May 10, 1938, had attained a fairly great size presenting thick stems with very short internodes, and large leaves. Toward that time they were forming buds. They looked by far better and more vigorous compared to the plants of blue-flowering *Lallemantia* sown in spring. Their stem was curved and thick, owing probably to their sparse distribution on the plot. The root system was well developed. Small plants also occurred among them and this might be interpreted to mean that the large plants had survived winter after being emerged from the seed fallen in the soil of the plot in autumn, while the smaller plants appeared above ground only in the spring and did not show therefore any considerable difference from the spring sown plants.

When flowers appeared on the plants (first flowering May 19—20), there was no doubt more that they all belonged to the blue-flowering form. And this in face of the fact that white- and pink-flowering forms had also been growing on that plot in the year 1937. Why had the latter not survived winter was not clear.

After treating the data on hibernation of the blue-flowering form of *Lallemantia* No. 90 and comparing them with those relating to the spring sown plants (unvernalized and vernalized at as low a temperature as 2° C) of this as well as the white- and pink-flowering forms, we obtained very interesting results.

Flowering set in a fortnight earlier in the hibernated plants compared to the unvernalized ones sown in spring as well as to white and pink forms of *Lallemantia*. With respect to the vernalized plants the difference is ranging within 5—9 days, according to the length of vernalization.

The yield data per plant are approximate, yet they testify unmistakably to the superiority of hibernated plants. The latter exceed not only the spring sown blue forms, especially the unvernalized plants, but also the large-seeded white and pink forms. The difference in absolute weight within the blue-flowering form is also in favour of hibernated plants. It is very small, however, and does not surpass the limits peculiar to this form. The unvernalized plants are the least in absolute weight, the vernalized ones occupy an intermediate position. The difference is measured by tenth of a gram.

In our further ([2]) as in our former experiments we invariably found the absolute weight of the blue form to be 2—2.5 times below that of white and pink ones. In respect of stem height the hybernated plants of *Lallemantia* exceed all others, not excepting the white-flowering ones, and this is especially conspicuous when they are compared to the unvernalized plants sown in spring. The length of the main (central) fruit cluster reflects the same relation. The interesting point to note here is that the average number of twigs per plant presents diametrically opposite figures: the hibernated plants of *Lallemantia* have the least number of twigs, though they are higher in yield than all the others. Within the blue-flowering form the number of twigs is related to vernalization. It is gradually increasing toward the unvernalized plants sown in spring. This corroborates the data obtained in our experiment on vernalization of *Lallemantia* carried out in the preceding year ([1]).

The yield of oil is not definitely related to the influence of low temperatures.

The comparison of the above data shows the hibernation to be responsible for a very wide shift in nearly all the indices considered which assume unfavourable values even if compared to those of the white and pink forms. This may be taken to mean that on hibernating the blue-flowering *Lallemantia* No. 90 undergoes a very long vernalization (throughout winter) and an enormous effect is thus created.

A weaker but also very remarkable influence is produced by pre-sowing treatment with low temperature for 30 days. Still less effective but also favourable is a short vernalization during 5 days (compared to the unvernalized control) which is sufficient to bring about important results.

Long vernalization at low temperature and especially hibernation have thus proved to produce highly valuable changes in blue-flowering *Lallemantia* (at least in the strain No. 90). And the longer the treatment, the better are the indices obtained.

With the aim to obtain new data in support of the above figures and to clear up the question as to the hibernation of white and pink-flowering forms of *Lallemantia*, we have started another experiment in the autumn of 1938 on the territory of the Botanical Garden in Odessa.

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PLANT PHYSIOLOGY

SECONDARY DORMANCY INDUCED IN SEEDS OF *LALLEMANTIA IBERICA* F. et M.

By D. J. VAKULIN

(Communicated by N. I. Vavilov, Member of the Academy, 13. IV. 1939)

It is known that the seeds of certain plants which have passed dormancy (after-ripening) or are nearly not-dormant at all (the dormant period being too short to be noticed) can return into resting condition under the influence of some or other factor. This phenomenon may appear in the seed either in the form of delayed and prolonged germination, or else the dormant state is so deep that it cannot be broken without great difficulty even if normal conditions of germination are secured.

Various causes may account for induced dormancy.

Crocker and Devis have revealed a similar phenomenon in the seed of *Alisma Plantago* treated with solutions of $CuSO_4$ (1). Devis observed it in the seeds of *Xanthium* (which are generally non-dormant) while keeping them at 30 ° C for 2—2.5 months under conditions which interfered with the admission of oxygen to the embryo.

Profound dormancy was induced by Kidd and West (2) in the swollen seed of white mustard under the action of the air enriched with carbon dioxide or of a certain mixture of CO_2 and O_2.

Poptzov has shown that thoroughly dried seeds of tobacco and makhorka *(Nicotiana rustica)* kept over a strong solution of sulphuric acid or dry NaOH for a period of 20 months germinate very slowly and germination is incomplete, especially in the case of freshly harvested seeds (3). The same author has observed secondary dormancy in the seeds of *Taraxacum megolorhizon* (Forsk) Rand-Maz, while using high and low temperatures (4).

Our own observations concerning the subject involve the blue-flowering form of *Lallemantia* No. 90.

Under the usual conditions of a field experiment the seeds of *Lallemantia* made fairly good and quick germination, but with changed situation, especially when grown in the light, they exhibited secondary dormancy. In our experiment we soaked the seeds of the above-mentioned form and on March 16, 1938, placed them on the window-sill in a Petri dish for germination in the light at room temperature. The immediate result of soaking was a very slimy condition of seeds. On observing them further we noticed that they were long failing to germinate. At first we ascribed this fact to the seeds having lost their germinating power, but as this proved to be not consistent with the truth, we sought for other responsible causes. More careful observation and check led us to the conclusion that light may possibly interfere

with successful germination. April 5, 1938, i. e. 20 days after the seeds were exposed for germination in the light, we took part of them out of the dish and planted these seeds into an earth-filled clay pot. The potted seeds were covered with earth, normal conditions of germination thus being secured after 20 days exposure to light. As early as 3 days later, i. e. April 8, we observed them to have produced seedlings, while the seeds left in the dish germinated neither on that day nor until April 16 and even 26.

The fact that one portion of seed was held in a dish, while the other in a pot filled with earth, suggested the idea that the medium, along with the light, may be a factor in germination. To verify that supposition another portion of ungerminated seed was taken from the Petri dish April 26 and its one half was potted under a thin layer of earth, while the second half was placed in another Petri dish and left to stand in the dark. Both in the dish and pot the seeds were wetted with water to keep them continually moist. Since the factor of light was then eliminated, there remained the sole difference in the medium (germination bed): the garden earth or the Petri dish. The seeds would not germinate for a long time and it was not until May 8—10, i. e. 12—14 days later, that very little germination could be observed both in the dish and pot, in the former somewhat weaker than in the latter. But no germination whatever appeared in the first Petri dish which remained in the light throughout the experiment (i. e. from March 16), though the seeds were being wetted there too and showed all signs of life.

There are two interesting points to note here. The first is that the longer the seeds were kept in the light for germination, the less successful the latter was after they had been transferred in the dark. The seeds held during 20 days in the light produced seedlings as soon as in three days after being covered with earth in the pot. Hence the period of 20 days failed to inhibit germination. Yet if germination in the dark (in a pot with earth or a Petri dish) was preceded by as long a period in the light as 40 days, it was only after a fortnight had passed that a few of the seeds began sprouting. The seeds which remained all the time in the light refused germination altogether. It will be obvious that while 20 days of light are not sufficient to inhibit germination in the dark, the drawing out of this period over 40 days brings about a state of such a deep dormancy that can be overcome only very slowly and with great difficulty.

The second point to note is that the light rather than the germination bed should be considered as the main responsible factor. This must not, however, be taken to mean that the latter is deprived of any influence on the seed. It does of course affect germination to a certain extent as it is known from several different experiments with other plants, but in our case the rôle played by the light has been of supreme significance. We have therefore to conclude that light is the factor inducing secondary dormancy in the seed of blue-flowering *Lallemantia*.

We did not content ourselves, however, with the data obtained and set up another experiment to verify the conclusion.

April 18 part of the seeds which refused germination in the Petri dish and had already dried owing to elimination of watering, were put in a bag of black paper and transferred to a dark room. The rest of the seeds, which were also dry, remained in the light (on the window-sill). So they were kept until May 18. Then a half of light-treated seeds and a half of dark-treated ones were put into separate dishes, soaked and left to germinate in the light. The other halves respectively treated were allowed to germinate in the dark.

The first two portions which were placed in the light failed to germinate. But the portions in the dark began germinating towards the end of May in

both dishes and if it was not the bulk of the seeds which did so, nevertheless, some of them came into growth.

This experiment goes to confirm the advanced supposition as to the responsible cause of the dormancy induced in *Lallemantia* seed. Here too, the factor of light has appeared to inhibit germination. In the light the seeds fall into dormancy of varying depth, while in the dark they come into growth and continue development. The indirect evidence furnished by the experiment is that the ungerminated seeds of *Lallemantia* No. 90 in spite of as long a period of preservation in the moist medium as 2 months do not lose their vitality and respond by germination to the advent of favourable conditions (darkness). Besides, we have to note here another interesting and important fact most forcibly brought out by the experiment. The seeds soaked, dried in the dish and soaked again a month later did not refuse and even made better germination in the dark than those which had not experienced drying. But whether dried or not they failed to germinate in the light.

The reported biological peculiarity of the blue-flowering *Lallemantia* is always to be taken into account while working with that plant, especially on determining the germinating powers of its seed material.

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PLANT PHYSIOLOGY

DETERMINATION OF BOTANICAL FORMS OF *LALLEMANTIA* FROM THE ABSOLUTE WEIGHT AND SIZE OF THEIR SEED

By D. J. VAKULIN

(Communicated by N. I. Vavilov, Member of the Academy, 13. IV. 1939)

While treating in the laboratory the seed of *Lallementia* harvested in 1936, 1937 and 1938, we found the different forms of this plant to be characterized by definite indices, particularly the absolute weight and size of their seed*. These relations are brought out in the table. In the first instance we see from the data referring to the year 1936 that there is a great difference between the white and the blue form with regard to the absolute weight of their seed: 5 g on the average in the former (4.346 to 5.391 g) as against 2.5 g in the latter (2.189 to 2.881 g), i. e. 2 times more. The difference in the absolute weight was quite manifest as early as in the first year of experiment. This difference has not disappeared from the yield after some of the samples harvested in 1936 were planted again in the subsequent years: 1937 and 1938. The quotient of the ratio between the respective figures in the table is again 2. This character is subject indeed to some fluctuations under the influence of changed environment and other conditions (origin of seed, etc.).

Obtained from	Absolute weight in g			Average length of seed, mm
	1936	1937	1938	
White and pink				
Leningrad	4.346	4.6803	4.6784	4.5—5.3
Zhitomir	4.931	—	4.7877	
Dnepropetrovsk	4.837	—	4.7810	
Rostov on the Don (No. 2)	—	—	5.2580	
Pensa	5.088	4.7106	4.9058	
Sophia	—	4.6383	4.6809	
Koimbra	4.894	—	—	
»	5.015	4.7905	4.8128	
Blue				
Moscow	2.189	2,2138	2.1519	3.0—3.8
Kiev	2.287	1.9805	2.2350	
Paris	2.333	—	—	
Madrid	2.881	2.3657	2,5166	

* The pink form is classed with the white one.

But the seasonal factors seem not to be able to efface such hereditary characters of these forms as the absolute weight and others which remain conspicuous enough. The weight of 1000 seeds (absolute weight) proves to be a very distinct peculiar feature of a given botanical form of *Lallemantia*, though the samples of the same form planted side-by-side may show individual variations.

No less definite character is the size of seed. It is evident from the table that the seeds are small in all the samples of the blue form, ranging in length between 3.0 and 3.8 mm. In the white-flowering form the length of seeds varies from 4.5 to 5.3 mm*. This applies to the three consecutive years: 1936, 1937, 1938. The results obtained in 1936 recurred in the subsequent years just as they did in the case of absolute weight. We have dwelt in the first instance upon these two characters, for they enable the worker in agriculture to decide whether the seeds to be planted belong to the blue or the white form of *Lallemantia*. This is so much the more important as our knowledge of the peculiar differences between different forms of that plant is very meagre and the seeds have been supplied to us under the common name of *Lallemantia iberica* or *Lallemantia* without regard to the form. Several difficulties and errors may be avoided if the determination or check of seed material with respect to botanical forms is based upon the absolute weight and size of seeds.

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* In some of the samples it even exceeded this figure.

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PALEONTOLOGY

A NEW GENUS *PARASHUMARDITES* AMONG UPPER CARBONIFEROUS AMMONITES OF NORTH AMERICA

By V. E. RUZHENCEV

(Communicated by A. A. Borissiak, Member of the Academy, 20. IV. 1939)

For a long time North America has been the only country where deposits of the Upper Carboniferous were known to contain a rich ammonite fauna. The absence of a similar faunal complex in other countries was no small hindrance for synchronizing corresponding sequences, and lead occasionally to serious misunderstandings. Some eminent American geologists argued, for instance, that deposits of the Upper Carboniferous were entirely absent in the Urals.

During recent years the author has collected in the southern part of the Urals a rather varied ammonite fauna, comprising characteristic genera of the Upper Carboniferous, such as *Uddenites*, *Vidrioceras*, *Shumardites*, *Schistoceras*, *Neodimorphoceras*, and others. This gives us ground to refute the contention of the American authors ([3],[4]). Besides, the treatment of the collections permits to settle or elucidate many obscure questions of evolution and taxonomy of this important group of fossils. In the present paper I should like to dwell upon the solution of some questions arising in connexion with the study of the genus *Shumardites*.

Five species are referred to this genus by American paleontologists, namely: *Sh. simondsi* Smith, *Sh. senex* Miller et Cline, *Sh. cuyleri* Plummer et Scott, *Sh. fornicatus* Plummer et Scott, and *Sh. sellardsi* Plummer et Scott. According to Smith ([5]), the genotype *Sh. simondsi* shows the following peculiarities in the structure of the suture line. Each part of the siphonal lobe is divided by an auxiliary saddle in two unequal digitations. The external lateral lobe consists of three sharply isolated parts which are regarded by Smith as three independent lateral lobes. The marginal two are bifid and the median one ends in one point, though with a tendency to become trifid. On the umbilical shoulder is located one more pointed lobe—the umbilical one. The antisiphonal lobe is tripartite. Besides the latter, Smith identifies on the internal wall of the conch two lateral lobes and one small auxiliary lobe along the umbilical seam. Unfortunately, the internal part of the suture line is not shown in Smith's figure.

Plummer and Scott ([2]) described *Sh. simondsi* Smith as possessing the following lobes: siphonal, first lateral sharply divided into three parts, second lateral extending near the umbilical shoulder, third lateral located on the umbilical wall, antisiphonal lobe, first internal lateral, and, finally, second internal lateral. The above authors likewise fail to give a picture of

the internal region, describing this part of the suture line, evidently, after Smith. Hence, even with this latest description on hand, we are still not sufficiently familiar with the structure of the suture line in *Sh. simondsi*. However, we have reason to assert that the description of the external region of the suture line given by Plummer and Scott, is at variance with the figures given by Smith (Table III, Fig. 12), as well as with their own drawing (Table 23, Fig. 1). These figures show very plainly that *Sh. simondsi* possesses in addition to the tripartite first lateral lobe but one more umbilical lobe. As to the internal part of the sutural line, it may be assumed by analogy with the material from the Urals that the three internal lobes in *Sh. simondsi*, arranged between the antisiphonal lobe and the umbilical seam, correspond to the one tripartite internal lateral lobe.

The analysis of the available material leads to the conclusion that the genotype of the genus *Shumardites* possesses 8 lobes about the curvature in the juvenile stage, and 16 at maturity. The Ural representatives fully answer these characteristics. Fig. 1 shows the development of the sutural line in *Shumardites confessus* sp. nov.

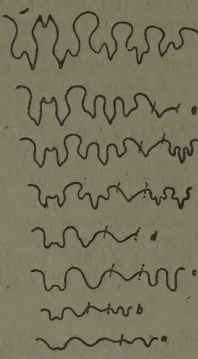

Fig. 1. Development of the sutural line in *Shumardites confessus* sp. nov. Specimens *a*—*e* and *g* were found on the right bank of the Ural river, 2.5 km north-east from Iljinskaja village (*a*—*e* figured from one specimen). Samples *f* and *h* found on the right bank of the Ural river, 2—3 km west of Nikolskaja village. Figured: *a*—measuring $W=0.9$ mm (×25); b—measuring $W=1.5$ mm, stage *Glaphyrites* (×12.5); *c*—with $W=2.7$ mm, terminal stage *Glaphyrites*(×12.5); *d*—with $W=3.3$ mm (×8); *e*—with $W=4.8$ mm; $H=2.2$ mm (×6); *f*—with $H=4.2$ mm (×5); *g*—with $W=8.5$ mm; $H=5.0$ mm, holotype (×5); *h*—measuring $W=19.0$ mm (×3). Upper Carboniferous.

Diagnosis. Conch rather involute, laterally somewhat compressed, in the siphonal region appreciably convex. Umbilicus of medium size. Sculpture presented by feebly developed transverse lirae. In the neanic stage the external wall bears 5 lobes, the internal one—three. Lateral and antisiphonal lobes are tripartite. In the mature stage each of the lateral lobes is transformed into three independent ones. Suture line of the holotype is shown on Fig. 1, *g*. Dimensions:

D	*H*	*W*	*Du*	*H/D*	*W/D*	*Du/D*
19.0	8.2	13.5	4.5	0.43	0.71	0.24

Remarks. Our species stands close to *Sh. simondsi* Smith, however, differing from the latter by higher organization. The width of the voluton measuring 19 mm shows a more differentiated suture line than the American species, the corresponding width being about 26 mm. In *Sh. confessus* the external digitation of medial branch of the lateral lobe is far more developed as compared to the internal one, whereas in *Sh. simondsi* the reverse is the case.

In the juvenile stage (Fig. 1, *e*, *f*) 8 lobes are observable. In adult specimens (Fig. 1, *h*) each of the lateral lobes is transformed into three independent lobes, which brings the total of all the lobes up to 16.

If we take other American representatives of the genus *Shumardites*, we note the appearance of entirely new characters. In *Sh. senex* Miller et Cline (Fig. 2) in quite mature specimens the branches of the siphonal lobe remain united ending in one point. The first lateral lobe consists of three complete marginal branches, their general outline fully conforming to the juvenile stage in *Sh. simondsi*. There is a second lateral lobe, rounded at the base. Umbilical lobe wedge-shaped. Antisiphonal lobe tripartite, as well as the first internal lateral one. There is a second internal lateral lobe—broad, rounded at the base. Briefly speaking, in *Sh. senex* 12 lobes are discernible about the curvature. Consequently, I consider *„Shumardites" senex* to be the genotype of the genus *Parashumardites* gen. nov.

D i a g n o s i s. Conch subglobose, rather involute, with the siphonal side strongly convex, umbilicus rather wide. The surface of the shell is covered with fine transverse traces. The suture line consists of 12 principal lobes and 12 saddles. The siphonal lobe is divided by a high siphonal saddle into two sharply pointed branches. The first lateral lobe is sharply tripartite; its marginal branches are curved and rounded, whereas the more elongated median subdivision is sharply pointed. The second lateral lobe has the same outline as the external branch of the first lateral lobe. The umbilical lobe is wedge-shaped. Antisiphonal lobe—tripartite. The first internal lateral lobe—asymmetrically tripartite. Second internal lateral lobe—rounded.

Genotype—«*Shumardites*» *senex* Miller et Cline (1934, p. 184, Table 28, Fig. 31—36). Nellie Bly formation, Oklahoma.

R e m a r k s. The new genus differs sharply from *Shumardites* in the number, arrangement and shape of lobes. At a casual glance it might look like a juvenile representative of *Shumardites*, though, in reality, it is distinguished from the latter by the presence of an additional pair of lateral lobes on each side of the conch.

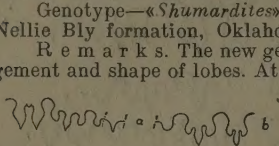

Fig. 2. *Parashumardites senex* (Miller and Cline), after Miller and Cline; *a*—external suture line (×2); *b*—internal suture line (×1.5). Nellie Bly formation, Oklahoma.

Miller and Cline [1] pointed out in their work that *Sh. simondsi* is a considerably more perfect species as compared to *Sh. senex*. This contention seems to be true only when the lateral lobes are compared. However, when comparison is made of complete suture lines, a different conclusion should be drawn. In *Shumardites* the complexity of the suture line advancing with growth of the shell was achieved at the expense of the increasing isolation of the branches of the lateral lobes. In *Parashumardites* the differentiation of the lateral lobes stops at a stage corresponding to the juvenile stage in *Shumardites*, but then, simultaneously, on each side of the conch appears a pair of new lobes arising evidently at the expense of the tripartite division of the umbilical lobe. Thus, comparing *Shumardites* and *Parashumardites* we see no reason for considering one of them superior to the other. Those genera had arisen during the Upper Carboniferous from a common ancestor which is to be identified as genus *Glaphyrites*; their development was parallel, each of them pursuing its own course.

The genus *Shumardites* includes *Sh. simondsi* Smith, *Sh. bakeri* (Plummer and Scott), *Sh. confessus* sp. nov. and *Sh. librovitchi* sp. nov. In the genus *Parashumardites* belong *P. senex* (Miller and Cline), *P. fornicatus* (Plummer and Scott), *P. cuyleri* (Plummer and Scott) and *P. sellardsi* (Plummer and Scott). All the enumerated forms have been met in Upper Carboniferous deposits.

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